Note on Cauchy-Riemann Equations

The integral of the complex function $f(z)$ is computed as a line integral along a curve C in the complex plane

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx+udy) \quad (1)$$

If this integral should be path independent, the expressions $udx - vdy$ and $vdx + udy$ must be total differentials

$$\begin{aligned} dw_1 &= udx - vdy \\ dw_2 &= vdx + udy \end{aligned} \quad (2)$$

Therefrom it follows with help of the chain rule

$$dw_1 = \frac{\partial w_1}{\partial x}dx + \frac{\partial w_1}{\partial y}dy \quad (3)$$

$$dw_2 = \frac{\partial w_2}{\partial x}dx + \frac{\partial w_2}{\partial y}dy \quad (4)$$

that

$$\frac{\partial w_1}{\partial x} = u, \frac{\partial w_1}{\partial y} = -v \quad (5)$$

$$\frac{\partial w_2}{\partial x} = v, \frac{\partial w_2}{\partial y} = u \quad (6)$$

and differentiating further

$$\frac{\partial w_1}{\partial x\partial y} = \frac{\partial u}{\partial y}, \frac{\partial w_1}{\partial y\partial x} = -\frac{\partial v}{\partial x} \quad (7)$$

$$\frac{\partial w_2}{\partial x\partial y} = \frac{\partial v}{\partial y}, \frac{\partial w_2}{\partial y\partial x} = \frac{\partial u}{\partial x} \quad (8)$$

we obtain the Cauchy-Riemann equations known also as Cauchy-Riemann conditions

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (9)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad (10)$$