Complex Integration

Cauchy's Theorem If $f(z)$ is an analytic function of z in a certain region including the point $z = a$, and if $\oint$ denotes the line integral along a closed contour within this domain taken around the point a in a counter-clockwise sense, then

$$\oint f(z)\,dz = 0 \tag{1}$$

and

$$\frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz = f(a) \tag{2}$$

From these two equations it is possible to derive the *theorem of residues,* which will now be stated.

Theorem of Residues Suppose that the function $f(z)$ can be expanded in the neighborhood of the point $z = z_0$ in the form

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

where m is some finite integer. If this expansion is possible, $f(z)$ is said to have a pole of order m at z_0. Then

$$\oint f(z)\,dz = 2\pi i a_{-1} \tag{3}$$

provided the integral is taken counter-clockwise about the point z_0. The coefficient a_{-1} is said to be the *residue* of the function $f(z)$. As a generalization of (3) we note that, if the contour of integration includes other poles at which the function has residues $b_{-1}, c_{-1}, \cdots$,

$$\oint f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \tag{4}$$