Notes on Correlation Function and Power Spectrum

From earlier derivations for the periodic function $x(t)$ of period T expanded in a Fourier series with real coefficients a_n and b_n we have the power ρ dissipated as

$$\rho = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{T}\sum_{n=0}^{\infty} w(f_n) \tag{1}$$

where

$$f_n = \frac{n}{T} \tag{2}$$

and

$$w(f_0) = T\frac{a_0^2}{4} \tag{3}$$

and for $n = 1, 2, \ldots$

$$w(f_n) = T\frac{1}{2}(a_n^2 + b_n^2) \tag{4}$$

$w(f_n)$ is the energy dissipated in a period T by the n-th harmonic component of frequency f_n. If

$$\Delta f_n = f_{n+1} - f_n = \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T} \tag{5}$$

then we may write

$$\rho = \sum_{n=0}^{\infty} w(f_n)\Delta f_n \tag{6}$$

and when $T \to \infty$ then

$$\rho = \int_0^{\infty} w(f)\,df \tag{7}$$

The function $w(f)$ has the dimension of energy and is called power spectrum. $w(f)\,df$ is the power dissipated by the current $x(t)$ in a unit resistance in the frequency range from f to $f + df$.

Also from earlier derivations for the periodic function $x(t)$ of period T expanded in a Fourier series with complex coefficients c_n we have the power ρ dissipated as

$$\rho = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{T}\sum_{n=-\infty}^{\infty} S(\omega_n) \tag{8}$$

where

$$\omega_n = \frac{2\pi n}{T} \tag{9}$$

and for $n = 0, \pm 1, \pm 2, \ldots$

$$S(\omega_n) = T c_n^* c_n \tag{10}$$

$S(\omega_n)$ is the energy dissipated in a period T by the n-th harmonic component of circular frequency ω_n. If

$$\Delta\omega_n = \omega_{n+1} - \omega_n = \frac{2\pi(n+1)}{T} - \frac{2\pi n}{T} = \frac{2\pi}{T} \tag{11}$$

then we may write

$$\rho = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} S(\omega_n)\Delta\omega_n \tag{12}$$

and when $T \to \infty$ then

$$\rho = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega \tag{13}$$

The function $S(\omega)$ has the dimension of energy and is called spectral density. $S(\omega)\, d\omega$ is the power dissipated by the current $x(t)$ in a unit resistance in the circular frequency range from ω to $\omega + d\omega$.

Now instead of power dissipated ρ let us have a look at the equation

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(t)x(t+\tau)\, dt \tag{14}$$

The above equation is the definition of the autocorrelation function $C(\tau)$ of the function $x(t)$. Expanding $x(t)$ in a Fourier series of exponential functions with imaginary exponents we have

$$x^*(t)x(t+\tau) = \Big(\sum_{m=-\infty}^{\infty} c_m e^{i\omega_m t}\Big)^* \Big(\sum_{n=-\infty}^{\infty} c_n e^{i\omega_n (t+\tau)}\Big) \tag{15}$$

$$= \Big(\sum_{m=-\infty}^{\infty} c_m^* e^{-i\omega_m t}\Big)\Big(\sum_{n=-\infty}^{\infty} c_n e^{i\omega_n (t+\tau)}\Big)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n e^{i(\omega_n - \omega_m)t} e^{i\omega_n \tau} \tag{16}$$

Then

$$
\begin{aligned}
C(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(t)x(t+\tau)\,dt && (17)\\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n e^{i(\omega_n-\omega_m)t} e^{i\omega_n\tau} \right) dt\\
&= \lim_{T\to\infty} \frac{1}{T} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n e^{i\omega_n\tau} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(\omega_n-\omega_m)t}\,dt\\
&= \lim_{T\to\infty} \frac{1}{T} \sum_{n=-\infty}^{\infty} T c_n^* c_n e^{i\omega_n\tau}\\
&= \lim_{T\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} S(\omega_n) e^{i\omega_n\tau} \Delta\omega_n\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau}\,d\omega && (18)
\end{aligned}
$$

The autocorrelation function

$$
C(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau}\,d\omega \tag{19}
$$

is the inverse Fourier transform for the spectral density $S(\omega)$ and

$$
S(\omega) = \int_{-\infty}^{\infty} C(\tau) e^{-i\omega\tau}\,d\tau \tag{20}
$$

the spectral density $S(\omega)$ is the direct Fourier transform of the autocorrelation function $C(\tau)$. The last equation is known as the Wiener-Khinchin theorem.