Notes on Power, Power Spectrum, and Correlation Function

(based on Kenneth S. Miller "Engineering Mathematics", pages:, 132, 164, Dover Publications 1963)

Suppose $x(t)$ is an arbitrary periodic function of period T. If we think of x as a current passing through a unit resistance, then the power ρ dissipated is

$$\rho = \frac{1}{T}\int_0^T x^2(t)\,dt \tag{1}$$

what is the mean square value of x. The above result is based on the formula for the power ρ dissipated by a DC current I passing through a resistance R

$$\rho = I^2 R \tag{2}$$

If we write the Fourier series expansion for $x(t)$,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n}{T}t + b_n \sin\frac{2\pi n}{T}t\right) \tag{3}$$

then

$$\rho = \frac{1}{T}\int_0^T \left[\frac{a_0^2}{4} + \left(a_n^2 \cos^2\frac{2\pi n}{T}t + b_n^2 \sin^2\frac{2\pi n}{T}t\right)\right] dt \tag{4}$$

since the integrals of all cross products $\sin 2\pi nt/T\ \cos 2\pi mt/T$, as well as the products $\sin 2\pi nt/T\ \sin 2\pi mt/T$ and $\cos 2\pi nt/T\ \cos 2\pi mt/T$ with $n \neq m$, vanish by virtue of the orthogonality conditions. Thus, since

$$\int_0^T \cos^2\frac{2\pi n}{T}t\,dt = \int_0^T \sin^2\frac{2\pi n}{T}t\,dt = \frac{T}{2} \tag{5}$$

we have

$$\rho = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{6}$$

and $\frac{1}{2}(a_n^2 + b_n^2)$ is the power content due to the n-th harmonic.

The period of the n-th harmonic is

$$\frac{T}{n} \tag{7}$$

The frequency of the n-th harmonic is

$$f_n = \frac{n}{T} \tag{8}$$

The energy $w(f_n)$ of the n-th harmonic per cycle is

$$w(f_n) = T\frac{1}{2}(a_n^2 + b_n^2) \tag{9}$$

Now we will extend the notion of power content to the non-periodic functions. Let us consider function of time, say $x(t)$, which extends from $-\infty$ to $+\infty$.

We define a new function $x^T(t)$ as follows

$$x^T(t) = x(t), \ -\frac{T}{2} < t \leq \frac{T}{2} \tag{10}$$

$$x^T(t+T) = x^T(t) \tag{11}$$

The function $x^T(t)$ is a periodic function of period T and using the properties of Fourier series with complex coefficients for the function $f(t)$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \tag{12}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-i\omega_n t}\,dt$$

where $n = 0, \pm 1, \pm 2, \ldots$ and

$$\omega_n = \frac{2\pi n}{T} \tag{13}$$

we may write for the function $x^T(t)$ with a slight change of notation

$$x^T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega_n)e^{i\omega_n t} \tag{14}$$

$$X(\omega_n) = \int_{-T/2}^{T/2} x^T(t)e^{-i\omega_n t}\,dt \tag{15}$$

For $T \to \infty$ we notice that

$$\lim_{T\to\infty} X(\omega_n) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega_n t}\,dt \tag{16}$$

We see that we arrive at the Fourier transform integral of the function $X(t)$. The full justification of the above derivation can be found in the chapter entitled ”Notes on Fourier Transform Derivation from Fourier Series of Exponential Functions with Imaginary Exponents”.

Suppose that $x(t)$ is a periodic function of period T. Then we may write $x(t)$ as a Fourier series, that is a sum of sinusoidal components of frequencies $0, f, 2f, \dots$, where $f = 1/T$. If we think of $x(t)$ as a current flowing through a unit resistance, it will dissipate a certain average amount of power, say ρ watts. Then for the frequency $f_n = nf$ the expression $w(f_n)/T$ gives the amount of watts dissipated from the current component of frequency f_n.

We have then

$$\rho = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{T}\sum_{n=0}^{\infty} w(f_n) \tag{17}$$

The function $w(f_n)$ has the dimension of energy and is called the power spectrum of $x(t)$. If we let

$$\Delta f_n = f_{n+1} - f_n = \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T} \tag{18}$$

then we receive

$$\rho = \sum_{n=0}^{\infty} w(f_n)\Delta f_n \tag{19}$$

If we let $T \to \infty$ then the discrete frequencies f_n tend toward a continuous variable f and the sum becomes an integral

$$\rho = \int_0^{\infty} w(f)\,df \tag{20}$$

Frequencies are non-negative. However if we take into account the circular frequencies ω_n and the Fourier series of exponential functions with imaginary exponents, the $n = 0, \pm 1, \pm 2, \dots$ and we may write

$$\rho = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} S(\omega_n)\Delta\omega_n \tag{21}$$

where $\Delta\omega_n = 2\pi/T$. For $T \to \infty$ the variable ω_n corresponds to the variable $-\infty < \omega < \infty$ and we have

$$\rho = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega \tag{22}$$

$S(\omega)$ also has the dimension of energy and it is called spectral density of $x(t)$.

The periodic function $x(t)$ of period T we may expand into a Fourier series of exponential functions with imaginary exponents

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \tag{23}$$

Then we have the following expression for $x^2(t)$

$$\begin{aligned} x^2(t) &= \Big(\sum_{m=-\infty}^{\infty} c_m e^{i\omega_m t}\Big)^* \Big(\sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}\Big) \\ &= \Big(\sum_{m=-\infty}^{\infty} c_m^* e^{-i\omega_m t}\Big) \Big(\sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}\Big) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n e^{i(\omega_n - \omega_m)t} \end{aligned} \tag{24}$$

Now we will use the above expression in the formula for the power dissipated. Using the orthogonality relations of the exponential functions with imaginary exponents

$$\int_0^T e^{i(n-m)\frac{2\pi}{T}t}\, dt = [t]_0^T = T \tag{25}$$

for $n = m$ and

$$\int_0^T e^{i(n-m)\frac{2\pi}{T}t}\, dt = \frac{T}{2(n-m)\pi i}[e^{i(n-m)\frac{2\pi}{T}t}]_0^T = 0 \tag{26}$$

for $n \neq m$ we receive the expression for power dissipated ρ as

$$\begin{aligned} \rho = \frac{1}{T}\int_0^T x^2(t)\, dt = \frac{1}{T}\int_0^T \Big(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n e^{i(\omega_n-\omega_m)t}\Big)\, dt \\ = \frac{1}{T} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n \int_0^T e^{i(\omega_n-\omega_m)t}\, dt \\ = \sum_{n=-\infty}^{\infty} c_n^* c_n \end{aligned} \tag{27}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-i\omega_n t}\, dt \tag{28}$$