On the vector triple product $\vec{U} \times (\vec{V} \times \vec{W})$

In the orthonormal system of coordinates we have the following relations for the versors $\hat{\imath}, \hat{\jmath}, \hat{k}$

$$
\begin{aligned}
\hat{\imath} \times \hat{\jmath} &= \hat{k} \\
\hat{\jmath} \times \hat{k} &= \hat{\imath} \\
\hat{k} \times \hat{\imath} &= \hat{\jmath} \\
\hat{\imath} \times \hat{\imath} &= 0 \\
\hat{\jmath} \times \hat{\jmath} &= 0 \\
\hat{k} \times \hat{k} &= 0
\end{aligned} \tag{1}
$$

In the vector triple product $\vec{U} \times (\vec{V} \times \vec{W})$ the vector $\vec{V} \times \vec{W}$ is normal to the plane of vectors $\vec{V}$ and $\vec{W}$. Therefore vector $\vec{U} \times (\vec{V} \times \vec{W})$ lies in the plane of vectors $\vec{V}$ and $\vec{W}$

$$\vec{U} \times (\vec{V} \times \vec{W}) = r\vec{V} + s\vec{W} \tag{2}$$

Let

$$\vec{V} = V\hat{\imath} \tag{3}$$

$$\vec{W} = a\hat{\imath} + b\hat{\jmath} \tag{4}$$

$$\vec{U} = c\hat{\imath} + d\hat{\jmath} + e\hat{k} \tag{5}$$

Then

$$\vec{V} \times \vec{W} = V\hat{\imath} \times (a\hat{\imath} + b\hat{\jmath}) = Va\hat{\imath} \times \hat{\imath} + Vb\hat{\imath} \times \hat{\jmath} = Vb\hat{k} \tag{6}$$

$$
\begin{aligned}
\vec{U} \times (\vec{V} \times \vec{W}) &= (c\hat{\imath} + d\hat{\jmath} + e\hat{k}) \times (Vb\hat{k}) \\
= Vbc\,\hat{\imath} \times \hat{k} + Vbd\,\hat{\jmath} \times \hat{k} + Vbe\,\hat{k} \times \hat{k} &= -Vbc\,\hat{\jmath} + Vbd\,\hat{\imath}
\end{aligned} \tag{7}
$$

We can write

$$(ac + bd)\vec{V} = (ac + bd)V\,\hat{\imath} = (\vec{U} \cdot \vec{W})\,\vec{V} = Vac\,\hat{\imath} + Vbd\,\hat{\imath} \tag{8}$$

$$(Vc)\vec{W} = (\vec{U} \cdot \vec{V})\vec{W} = (Vc)(a\hat{\imath} + b\hat{\jmath}) = Vac\,\hat{\imath} + Vbc\,\hat{\jmath} \tag{9}$$

Now from equation (8) we subtract equation (9) obtaining

$$\vec{U} \times (\vec{V} \times \vec{W}) = (\vec{U} \cdot \vec{W})\,\vec{V} - (\vec{U} \cdot \vec{V})\vec{W} \tag{10}$$

References

[1] Lindgren, B.W. (1963) *Vector Calculus* The Macmillan Company, New York, Collier-Macmillan Limited, London

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