Notes on Taylor and Maclaurin Series Derivation

Let us assume that for the function $f(x)$ the following power series expansion is possible

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \dots \tag{1}$$

We see that if we set $x = a$ then

$$A_0 = f(a) \tag{2}$$

If we compute the first derivative of the series expansion we receive

$$f'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + 4A_4(x-a)^3 + \dots \tag{3}$$

and substitute $x = a$ then

$$A_1 = f'(a) \tag{4}$$

If we compute the second derivative of the series expansion we get

$$f''(x) = 2A_2 + 2\cdot 3A_3(x-a) + 3\cdot 4A_4(x-a)^2 + 4\cdot 5A_5(x-a)^3 + \dots \tag{5}$$

After substituting $x = a$

$$A_2 = \frac{f''(a)}{2} \tag{6}$$

Again if we compute the third derivative of the series expansion

$$\begin{aligned} f^{(3)}(x) = 2\cdot 3A_3 + 2\cdot 3\cdot 4A_4(x-a) + 3\cdot 4\cdot 5A_5(x-a)^2 \\ + 4\cdot 5\cdot 6A_6(x-a)^3 + \dots \end{aligned} \tag{7}$$

and substitute $x = a$ we receive

$$A_3 = \frac{f^{(3)}(a)}{2\cdot 3} \tag{8}$$

Computing higher derivatives of $f(x)$ it can be seen that the expression for the coefficient A_n in the power series is

$$A_n = \frac{f^{(n)}(a)}{n!} \tag{9}$$

Using the above result we see that the original power series expansion is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{10}$$

The above expression is called the Taylor series of the function $f(x)$. If in the Taylor series $a = 0$ we receive the Maclaurin series of $f(x)$.

Example: Taylor and Maclaurin Series of e^x

For $f(x) = e^x$ we have the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n \tag{11}$$

because all derivatives of e^x are just e^x. Substituting $a = 0$ in the Taylor series we receve the Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \tag{12}$$

Example: Taylor and Maclaurin Series of $\sin(x)$

Let us assume that the Taylor expansion of $\sin(x)$ is possible

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{13}$$

We compute the derivatives of $\sin(x)$ as follows

$$f^{(0)}(a) = \sin(a) \tag{14}$$

$$f^{(1)}(a) = \cos(a) \tag{15}$$

$$f^{(2)}(a) = -\sin(a) \tag{16}$$

$$f^{(3)}(a) = -\cos(a) \tag{17}$$

$$f^{(4)}(a) = \sin(a) \tag{18}$$

We notice that after the derivative $f^{(3)}(a)$ the next derivative is equal to the derivative $f^{(0)}(a)$ and the pattern is being repeated.

The Taylor expansion of $\sin(x)$ then is

$$\begin{aligned} \sin(x) = \frac{\sin(a)}{0!} + \frac{\cos(a)}{1!}(x-a) - \frac{\sin(a)}{2!}(x-a)^2 \\ -\frac{\cos(a)}{3!}(x-a)^3 + \frac{\sin(a)}{4!}(x-a)^4 + \frac{\cos(a)}{5!}(x-a)^5 \\ -\frac{\sin(a)}{6!}(x-a)^6 - \frac{\cos(a)}{7!}(x-a)^7 + \dots \end{aligned} \tag{19}$$

If we substitute $a = 0$ we receive Maclaurin series

$$\begin{aligned}\sin(x) &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\end{aligned} \tag{20}$$

Example: Taylor and Maclaurin Series of $\cos(x)$

Let us assume that the Taylor expansion of $\cos(x)$ is possible

$$\cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{21}$$

We compute the derivatives of $\cos(x)$ as follows

$$f^{(0)}(a) = \cos(a) \tag{22}$$

$$f^{(1)}(a) = -\sin(a) \tag{23}$$

$$f^{(2)}(a) = -\cos(a) \tag{24}$$

$$f^{(3)}(a) = \sin(a) \tag{25}$$

$$f^{(4)}(a) = \cos(a) \tag{26}$$

We notice that after the derivative $f^{(3)}(a)$ the next derivative is equal to the derivative $f^{(0)}(a)$ and the pattern is being repeated.

The Taylor expansion of $\cos(x)$ then is

$$\begin{aligned}\cos(x) &= \frac{\cos(a)}{0!} - \frac{\sin(a)}{1!}(x-a) - \frac{\cos(a)}{2!}(x-a)^2 \\ &+ \frac{\sin(a)}{3!}(x-a)^3 + \frac{\cos(a)}{4!}(x-a)^4 - \frac{\sin(a)}{5!}(x-a)^5 \\ &- \frac{\cos(a)}{6!}(x-a)^6 + \frac{\sin(a)}{7!}(x-a)^7 + \dots\end{aligned} \tag{27}$$

If we substitute $a = 0$ we receive Maclaurin series

$$\begin{aligned}\cos(x) &= \frac{1}{0!} - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\end{aligned} \tag{28}$$

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