Square Pyramidal Number and Polynomials

Pawel Jan Piskorz

The square pyramidal number represents the number of stacked spheres in a pyramid with a square base. The number of spheres in each layer of the pyramid is just a squared number of the length of the layer edge, and the square pyramidal number is the sum of the numbers of spheres in each layer. The first few square pyramidal numbers are: 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240 appropriately for the sum of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 square pyramid layers.

It follows from the definition of square pyramidal number $P(N)$ where N is the number of layers of spheres in the square base pyramid that the square pyramidal number is equal to the sum of the squares of the consecutive natural numbers

$$P(N) = \sum_{k=1}^{N} k^2 \tag{1}$$

In [1] the author presented a method of computing square pyramidal number as a function of the number N of the layers of square pyramid using the properties of the multiplication table. In this article we will present a simpler method of computing the square pyramidal number suggested in [2] in a footnote on page 81. Authors notice that

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1 \tag{2}$$

We may subtract k^3 from the left side of the equation (2) and we receive

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1 \tag{3}$$

We did that just to leave on the right side the term with k^2. We can write for $k = 1, \dots, N$

$$\begin{array}{rlcl}
k=1: & 2^3 - 1^3 & = & 3\cdot 1^2 + 3\cdot 1 + 1 \\
k=2: & 3^3 - 2^3 & = & 3\cdot 2^2 + 3\cdot 2 + 1 \\
k=3: & 4^3 - 3^3 & = & 3\cdot 3^2 + 3\cdot 3 + 1 \\
k=4: & 5^3 - 4^3 & = & 3\cdot 4^2 + 3\cdot 4 + 1 \\
\dots & & & \\
k=m-1: & m^3 - (m-1)^3 & = & 3\cdot (m-1)^2 + 3\cdot (m-1) + 1 \\
k=m: & (m+1)^3 - m^3 & = & 3\cdot m^2 + 3\cdot m + 1 \\
\dots & & & \\
k=N-1: & N^3 - (N-1)^3 & = & 3\cdot (N-1)^2 + 3\cdot (N-1) + 1 \\
k=N: & (N+1)^3 - N^3 & = & 3\cdot N^2 + 3\cdot N + 1
\end{array} \tag{4}$$

If we add the equations (4) we receive

$$(N+1)^3 - 1 = 3\sum_{k=1}^{N} k^2 + 3\sum_{k=1}^{N} k + N \tag{5}$$

We compute

$$N^3 + 3N^2 + 3N = 3\sum_{k=1}^{N} k^2 + 3\sum_{k=1}^{N} k + N \tag{6}$$

Because

$$\sum_{k=1}^{N} k = N(N+1)/2 \tag{7}$$

(see Appendix A) we have

$$N^3 + 3N^2 + 2N = 3\sum_{k=1}^{N} k^2 + 3N(N+1)/2 \tag{8}$$

$$N^3 + 3N^2 + 2N = 3\sum_{k=1}^{N} k^2 + 3N^2/2 + 3N/2 \tag{9}$$

$$2N^3 + 6N^2 + 4N = 6\sum_{k=1}^{N} k^2 + 3N^2 + 3N \tag{10}$$

$$2N^3 + 3N^2 + N = 6\sum_{k=1}^{N} k^2 \tag{11}$$

$$N(2N^2 + 3N + 1) = 6\sum_{k=1}^{N} k^2 \tag{12}$$

On the left side of the equation (12) we notice that we may solve quadratic equation $2N^2 + 3N + 1 = 0$ for N. The discriminant Δ of this quadratic equation $aN^2 + bN + c$ is equal to

$$\Delta = b^2 - 4ac = 9 - 4 \cdot 2 \cdot 1 = 1 \tag{13}$$

and the roots N_1 and N_2 of the quadratic equation $aN^2 + bN + c = a(N - N_1)(N - N_2)$ are

$$N_1 = \frac{-b - \sqrt{\Delta}}{2a} = -1 \tag{14}$$

$$N_2 = \frac{-b + \sqrt{\Delta}}{2a} = -\frac{1}{2} \tag{15}$$

We receive then

$$6\sum_{k=1}^{N} k^2 = N[2(N+1)(N+\frac{1}{2})] \tag{16}$$

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6} \tag{17}$$

We notice that similarly one can work out the formulas for the sum of cubes, and higher powers of consecutive natural numbers. For example, looking at the Pascal triangle (typesetting in LaTeX inspired by [3])

$$\begin{array}{lllllllllll}
n=0: & (k+1)^0 & & & & & 1 & & & & \\
n=1: & (k+1)^1 & & & & 1 & & 1 & & & \\
n=2: & (k+1)^2 & & & 1 & & 2 & & 1 & & \\
n=3: & (k+1)^3 & & 1 & & 3 & & 3 & & 1 & \\
n=4: & (k+1)^4 & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \tag{18}$$

we quickly compute

$$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1 \tag{19}$$

and again we may use

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1 \tag{20}$$

and take advantage of the above equation writing it N times for $k = 1, \ldots, N$ each, and adding the left and right side of the resulting equations receiving in this way after computations the formula for the sum of cubes of consecutive natural numbers.

Appendix A Sum of natural numbers

Let us focus on consecutive natural numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and find a faster method to compute their sum than just adding all the numbers together one by one. Let us write the following two and corresponding sequences

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \tag{21}$$

We notice that the sum of each two numbers occuring one under another is equal to 11. It is true for the sum of 1 and 10. We then move to the right to the next two numbers and we notice that the upper number is increased by 1 and the lower number is decreased by 1 and therefore their sum is the same as for the two appropriate numbers next to the left. So, if we multiply 11 by ten, we receive the sum of consecutive numbers from 1 to 10 taken twice. Therefore the sum of consecutive natural numbers from 1 to 10 is equal

$$(1 + 10) \cdot 10/2 = 55 \tag{22}$$

We see that to compute the sum of numbers 1 to 10 we need to add the first number to the last one, multiply the result by the number of the numbers in the sequence of numbers we want to add, and divide the result by two. In general for the natural numbers $1, 2, 3, \ldots, N$ we can write

$$\begin{array}{cccccccccc} 1 & 2 & 3 & \ldots & m & \ldots & N-2 & N-1 & N \\ N & N-1 & N-2 & \ldots & n & \ldots & 3 & 2 & 1 \end{array} \tag{23}$$

We see that the sum of a number from the upper sequence and the corresponding number just below it from the lower sequence is always equal to $N+1$, so for any number m from the upper sequence and the corresponding number n from the lower sequence we always have $m+n = N+1$. If we multiply $N+1$ by the number N, we receive the doubled sum of natural numbers $1, 2, 3, \ldots, N$

$$2(1+2+\ldots+N) = (N+1)N = N(N+1) \tag{24}$$

and therefore

$$1+2+\ldots+N = \frac{N(N+1)}{2} \tag{25}$$

Using the symbol Σ of the sum we denote

$$1+2+\ldots+N = \sum_{k=1}^{N} k \tag{26}$$

as the sum of numbers k having values from 1 to N, and we receive the result

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2} \tag{27}$$

The above formula is most probably due to Carl Friedrich Gauss. I learned the formula in equation (27) and the way of deriving it described here from my father Stefan Piskorz.

References

[1] Pawel Jan Piskorz (2016) *100.02 Square pyramidal numbers and the multiplication table* The Mathematical Gazette, Volume 100, Issue 547, March 2016, pp. 108-111, doi:10.1017/mag.2016.11.

[2] A.D. Aleksandrov, A.N. Kolmogorov, M.A. Lavrent'ev (1990) *Mathematics, Its Content, Methods, and Meaning* Volume One, Dorset Press, New York

[3] http://www.bedroomlan.org/coding/pascals-triangle-latex