Error Function, Dirichlet Kernel and the Improper Integral of Sin(x)/x

The function $\sin(x)/x$ where $x \in R$ does not have a known antiderivative and the improper integral of $\sin(x)/x$ in 0 to ∞ cannot be computed by subtracting the appropriate antiderivative limits at ∞ and at 0.

In this article we compute the integral

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin(x)}{x}\, dx \tag{1}$$

using two interesting methods. The first method leads us to the result involving the error function $\mathrm{erf}(x)$ the properties of which we use, and the second method uses the expression known as the Dirichlet kernel on which we perform some operations.

Method using the error function The method involving the error function was considered by the author as a result of taking into account a possibility of solving the improper integral I as a continuous function of μ

$$I(\mu) = \int_{-\infty}^\infty \frac{\sin(\mu x)}{x} e^{-ax^2}\, dx \tag{2}$$

The author however did not know the solution of this integral at the beginning and used [1] to see it first, and then worked out the method of derivation of the solution manually. There was no indication in [1] how the solution of the integral in equation (2) was obtained.

We compute the partial derivative with respect to the μ variable

$$\frac{\partial I(\mu)}{\partial \mu} = \int_{-\infty}^\infty \frac{x\cos(\mu x)}{x} e^{-ax^2}\, dx = \int_{-\infty}^\infty \cos(\mu x) e^{-ax^2}\, dx \tag{3}$$

The integral

$$\int_{-\infty}^\infty \cos(\mu x) e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}} e^{-\mu^2/4a} \tag{4}$$

has been derived in [2] on pages 309-310.

Now it is time then to solve the equation

$$\frac{\partial I(\mu)}{\partial \mu} = \sqrt{\frac{\pi}{a}} e^{-\mu^2/4a} \tag{5}$$

Let us consider two values of the integral $I(\mu)$ when $\mu = 0$ and when $\mu = 1$. We have the following situations which can be noticed just by substituting the above values of μ into the integral $I(\mu)$ in the equation (2). Then we have

$$I(\mu = 0) = 0$$

$$I(\mu = 1) = \int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{-ax^2}\, dx$$

Our intermediate result $I(\mu = 1)$ can become computed by assigning the $\mu = 0$ and $\mu = 1$ limits of integration for the integral I after integrating the equation (5)

$$I = \sqrt{\frac{\pi}{a}} \int e^{-\mu^2/4a}\, d\mu \tag{6}$$

We receive the expression $I(\mu = 1) - I(\mu = 0) = I(\mu = 1)$ by computing the definite integral

$$I(\mu = 1) = \int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}} \int_0^1 e^{-\mu^2/4a}\, d\mu \tag{7}$$

We substitute $\mu^2/4a = t^2$ and we receive

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{-ax^2}\, dx = 2\sqrt{\pi} \int_0^{1/2\sqrt{a}} e^{-t^2}\, dt \tag{8}$$

$$= 2\sqrt{\pi}\frac{1}{2}\sqrt{\pi}\left(\frac{2}{\sqrt{\pi}} \int_0^{1/2\sqrt{a}} e^{-t^2}\, dt\right) = \pi \operatorname{erf}\left(\frac{1}{2\sqrt{a}}\right)$$

We have used the error function $\operatorname{erf}(x)$ which is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{9}$$

The error function $\operatorname{erf}(x)$ approaches the value of 1 if $x \to \infty$. We may then write

$$\lim_{a \to 0} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{-ax^2}\, dx = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \lim_{a \to 0} \pi \operatorname{erf}\left(\frac{1}{2\sqrt{a}}\right) = \pi \tag{10}$$

We have then arrived at our main result applying the method using the error function.

Method using the Dirichlet kernel The expression

$$\frac{\sin\left[(2n+1)\theta/2\right]}{\sin(\theta/2)} \tag{11}$$

is known in mathematics as the Dirichlet kernel. It occurs in the sum of cosines $\frac{1}{2}+\sum_{k=1}^{n}\cos(k\theta)$. Such sum has been derived in [3] and modified for this article, and it is also equivalent to the one proven in [2]

$$\frac{1}{2}+\sum_{k=1}^{n}\cos(k\theta)=\frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)} \tag{12}$$

The author was not aware that the use of the sum of cosines to compute the improper integral of $\sin(x)/x$ was proposed in problem 18-55 in [4]. Integrating both sides of the above equation over θ in $[-\pi,\pi]$ we have

$$\int_{-\pi}^{\pi}\frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}\,d\theta=\pi \tag{13}$$

because the cosines integrals equal zero. Let us substitute $\theta = x/n$, $d\theta = dx/n$ for $n = 1, 2, \ldots$

$$\int_{-\pi}^{\pi}\frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}\,d\theta=\int_{-n\pi}^{n\pi}\frac{\sin\left[(2n+1)x/2n\right]}{2n\sin(x/2n)}\,dx \tag{14}$$

In particular, for $n\to\infty$

$$\lim_{n\to\infty}\int_{-n\pi}^{n\pi}\frac{\sin\left[(2n+1)x/2n\right]}{2n\sin(x/2n)}\,dx=\int_{-\infty}^{\infty}\frac{\sin(x)}{x}\,dx=\pi \tag{15}$$

We have computed separately the limits of the numerator and the denominator. We show only for the denominator that

$$\lim_{n\to\infty}2n\sin(x/2n)=\lim_{n\to\infty}x\frac{\sin(x/2n)}{x/2n}=x\lim_{\alpha\to 0}\frac{\sin(\alpha)}{\alpha}=x$$

This concludes our two different derivations of the improper integral of $\sin(x)/x$. For the computations involving the theory of residues we refer the reader to the book [5].

References

[1] *http://www.wolframalpha.com/*

[2] G.P. Tolstov, *Fourier Series*, Translation by R.A. Silverman, Dover Publications, Inc., New York NY, 1976.

[3] K.E. Hirst, *Numbers, Sequences and Series*, Arnold, London, Sydney, Auckland, 1995.

[4] M. Spivak, *Calculus*, Publish or Perish, Inc., Houston TX, 1994.

[5] E.B. Saff, A.D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, Prentice Hall, Upper Saddle River NJ, 2003.

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