# Note on matrix of rotation and multiplication of rotation matrices

**Abstract**

In this note the matrix of rotation by the angle $\alpha + \beta$ is derived, and formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ are determined. The origin of matrix multiplication is presented.

In the equation

$$\begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{1}$$

the expression

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{2}$$

is called the matrix of rotation of the vector

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{3}$$

by angle $\alpha$ about the $Oz$ axis counterclockwise. Rotated vector has the coordinates

$$\begin{bmatrix} v'_x \\ v'_y \end{bmatrix} \tag{4}$$

which are computed as

$$\begin{aligned} v'_x &= v_x \cos\alpha - v_y \sin\alpha \\ v'_y &= v_x \sin\alpha + v_y \cos\alpha \end{aligned} \tag{5}$$

Equations (1) and (5) are equivalent. The expressions in equations (5) are just the results of multiplication of the vector (3) by the matrix (2). As the result of such multiplication of the vector (3) by the matrix (2) we receive a column vector (6)

$$\begin{bmatrix} v_x \cos\alpha - v_y \sin\alpha \\ v_x \sin\alpha + v_y \cos\alpha \end{bmatrix} \tag{6}$$

The column vector (6) has the coordinates of the vector (3) which is rotated counterclockwise by the angle $\alpha$ around the $Oz$ axis.

Let us now apply the matrix of counterclockwise rotation around the axis $Oz$ by the angle $\beta$ on the coordinates of the column vector (6)

$$\begin{aligned}
\begin{bmatrix} v''_x \\ v''_y \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \begin{bmatrix} \cos\beta v'_x - \sin\beta v'_y \\ \sin\beta v'_x + \cos\beta v'_y \end{bmatrix} = \qquad (7)\\
= \begin{bmatrix} \cos\beta(v_x \cos\alpha - v_y \sin\alpha) - \sin\beta(v_x \sin\alpha + v_y \cos\alpha) \\ \sin\beta(v_x \cos\alpha - v_y \sin\alpha) + \cos\beta(v_x \sin\alpha + v_y \cos\alpha) \end{bmatrix} \\
= \begin{bmatrix} v_x(\cos\alpha\cos\beta - \sin\alpha\sin\beta) - v_y(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ v_x(\sin\alpha\cos\beta + \cos\alpha\sin\beta) + v_y(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \end{bmatrix} \\
= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \\
= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}
\end{aligned}$$

We can write

$$\begin{aligned}
\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \qquad (8)\\
= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix} \\
= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}
\end{aligned}$$

By comparison from equation (8) we see that we receive the formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ as

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{9}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{10}$$

If we write

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{11}$$

and

$$B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \tag{12}$$

we can compute the product of matrices $BA = C$

$$C = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix} \tag{13}$$

We notice that if $a_{i,j}, b_{i,j}, c_{i,j}$ denote the matrix element from $i$-th row and $j$-th column from the matrix $A, B, C$, respectively, we may write

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \tag{14}$$

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \tag{15}$$

and

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} \tag{16}$$

and we have for the matrix multiplication of the matrices $AB$ equals $C$ when A, B have the dimensions two rows by two columns

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} \tag{17}$$

In general for the matrices of dimensions $N$ by $N$ we have

$$c_{i,j} = \sum_{k=1}^{N} a_{i,k}b_{k,j} \tag{18}$$

We notice that when $A$ is the rotation matrix by the angle $\alpha$ and $B$ is the rotation matrix by the angle $\beta$, both rotations around the $Oz$ axis, we have

$$AB = BA \tag{19}$$

because the order of rotations does not matter, and first we may rotate by the angle $\alpha$ then by the angle $\beta$ or in the reversed order and we will receive the same result of rotations. In case of such rotations the matrix multiplication is commutative however the matrix multiplication is not commutative in general and for any matrices $A$, $B$

$$AB \neq BA \tag{20}$$

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