Note on matrix of rotation

Matrix of rotation in the x, y-plane We rotate the system of coordinates x, y by the angle α into the system of coordinaes x', y'. Then the versors (unit vectors) $\hat{i}$ and $\hat{j}$ are rotated by the angle α into new versors $\hat{i}'$ and $\hat{j}'$ (see Figure 1).

We express the new versor $\hat{i}'$ in the coordinates x, y as

$$\hat{i}' = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j} \tag{1}$$

because the value of the projection of the versor $\hat{i}'$ onto the Ox axis is $\cos\alpha$ and the value of the projection of the versor $\hat{i}'$ onto the Oy axis is $\sin\alpha$. Similarly, we have the following relation for the versor $\hat{j}'$

$$\hat{j}' = -\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j} \tag{2}$$

because the value of the projection of the versor $\hat{j}'$ onto the Ox axis is $-\sin\alpha$ and the value of the projection of the versor $\hat{j}'$ onto the Oy axis is $\cos\alpha$.

Any vector $\vec{v}$ can be expressed in coordinates Oxy

$$\vec{v} = v_x\,\hat{i} + v_y\,\hat{j} \tag{3}$$

or in coordinates $Ox'y'$ as

$$\vec{v} = v_{x'}\,\hat{i}' + v_{y'}\,\hat{j}' \tag{4}$$

We can rewrite equation (4) inserting the expressions for $\hat{i}'$ from the equation (1) and for $\hat{j}'$ from the equation (2)

$$\begin{aligned} \vec{v} &= v_{x'}(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}) + v_{y'}(-\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j}) \\ &= (v_{x'}\cos\alpha - v_{y'}\sin\alpha)\hat{i} + (v_{x'}\sin\alpha + v_{y'}\cos\alpha)\hat{j} \end{aligned} \tag{5}$$

The equation (5) expresses the vector $\vec{v}$ with primed coordinates $v_{x'}$ and $v_{y'}$ in original unrotated coordinate system Oxy. Comparing equation (5) with equation (3) we see that

$$\begin{aligned} v_{x'} \cos\alpha - v_{y'} \sin\alpha &= v_x \\ v_{x'} \sin\alpha + v_{y'} \cos\alpha &= v_y \end{aligned} \tag{6}$$

We may solve the system of linear equations (6) to calculate $v_{x'}$ and $v_{x'}$, i.e. the coordinates of the vector $\vec{v}$ in the rotated coordinate system $Ox'y'$. The determinants D, $D_{v_{x'}}$, $D_{v_{y'}}$ are

$$D = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1 \tag{7}$$

$$D_{v_{x'}} = \begin{vmatrix} v_x & -\sin\alpha \\ v_y & \cos\alpha \end{vmatrix} = v_x \cos\alpha + v_y \sin\alpha \tag{8}$$

$$D_{v_{y'}} = \begin{vmatrix} \cos\alpha & v_x \\ \sin\alpha & v_y \end{vmatrix} = -v_x \sin\alpha + v_y \cos\alpha \tag{9}$$

and therefore using the formulas $v_{x'} = D_{v_{x'}}/D$ and $v_{y'} = D_{v_{y'}}/D$ we obtain

$$\begin{aligned} v_{x'} &= v_x \cos\alpha + v_y \sin\alpha \\ v_{y'} &= -v_x \sin\alpha + v_y \cos\alpha \end{aligned} \tag{10}$$

Applying the matrix notation we have

$$\begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{11}$$

The above matrix equation (11) gives the coordinates of original vector $\vec{v}$ in rotated system $Ox'y'$. So it is as if the vector in the primed system $Ox'y'$ is rotated by the angle $-\alpha$. Then the coordinates v'_x and v'_y of the vector as rotated by angle α will be

$$\begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{12}$$

if the vector had original coordinates v_x and v_y because we had to change the sign by angle α to the opposite one. The matrix in the equation (12) is the rotation matrix of a point of coordinates v_x and v_y by the angle α.

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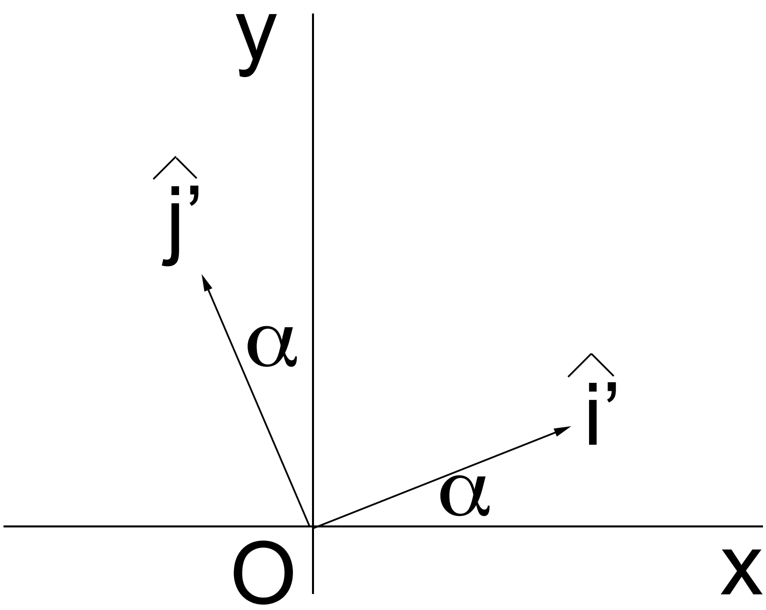

Figure 1: We rotate the system of coordinates x, y by the angle α into the system of coordinaes x', y'. Then the versors (unit vectors) $\hat{i}$ and $\hat{j}$ are rotated by the angle α into new versors $\hat{i}'$ and $\hat{j}'$.