Note on the solution of quadratic equation

We derive the solution of quadratic equation $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}, a \neq 0$

$$ax^2 + bx + c = a\left(x^2 + x\frac{b}{a} + \frac{c}{a}\right) \tag{1}$$
$$= a\left(x^2 + 2x\frac{b}{2a} + \frac{c}{a}\right) = a\left(x^2 + 2x\frac{b}{2a} + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

We notice in (1) the pattern

$$A^2 + 2AB + B^2 = (A + B)^2 \tag{2}$$

for

$$\left(x + \frac{b}{2a}\right)^2 = \left(x^2 + 2x\frac{b}{2a} + \frac{b^2}{4a^2}\right) \tag{3}$$

$$ax^2 + bx + c = a\left(x^2 + 2x\frac{b}{2a} + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right) \tag{4}$$
$$= a\left[\left(x^2 + 2x\frac{b}{2a} + \frac{b^2}{4a^2}\right) - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right)\right]$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{4ac}{4a^2}\right)\right]$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right]$$

In the last expression in (4) we notice the pattern

$$A^2 - B^2 = (A + B)(A - B) \tag{5}$$

and after rewriting (4) appropriately we receive

$$ax^2 + bx + c = a\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) \tag{6}$$

Rewriting further we obtain

$$ax^2 + bx + c = a\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \tag{7}$$

The expression $b^2 - 4ac$ for the quadratic equation $ax^2 + bx + c$ is known as the discriminant of the quadratic equation and it is being denoted as Δ

$$\Delta = b^2 - 4ac \tag{8}$$

We have then

$$\begin{aligned} ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{\Delta}}{2a}\right)\left(x - \frac{-b - \sqrt{\Delta}}{2a}\right) \\ = a(x - x_1)(x - x_2) \end{aligned} \tag{9}$$

and x_1, x_2 are being called the roots of the quadratic equation. We see that for $\Delta = 0$ the roots x_1, x_2 are equal

$$x_1 = x_2 = \frac{-b}{2a} \tag{10}$$

and we have one solution called a double root of the quadratic equation. For the $\Delta > 0$ we have two diffrent roots

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a} \tag{11}$$

$$x_2 = \frac{-b - \sqrt{\Delta}}{2a} \tag{12}$$

For $\Delta < 0$ we have two diffrent imaginary roots as the solution of the quadratic equation $ax^2 + bx + c$ because we need to compute a square root of a negative number and we need to use imaginary units in the appropriate calculation.

Pawel J. Piskorz (paweljs@gmail.com)