

Note on permutations

Let us take into account permutations of elements which are natural numbers. For example the permutation 3241 contains four transpositions: 32, 31, 21, 41 and the permutation 2341 contains three transpositions: 21, 31, 41.

To count the transpositions of a given permutation one can take into account the transpositions of the first element of the permutation, then the transpositions of the second element of the permutation, and so on.

There are permutations with odd or even transpositions of elements. These permutations are called odd or even permutations, respectively. In other words there are two classes of permutations: class of odd permutations and class of even permutations.

For example the permutation 3241 is even and it belongs to the class of even permutations, and the permutation 2341 is odd and it belongs to the class of odd permutations.

Theorem 1. At the transposition of any two of not necessarily contiguous elements the permutation changes its class.

Let us assume namely that two elements a_l and a_p in a permutation are separated by k other elements

$$\dots a_l a_{l+1} \dots a_{l+k} a_p \dots \quad (1)$$

In order to move the element a_l to the right just behind the element a_p one needs to interchange the element a_l with the element next to it on the right $k + 1$ times. Next, in order to put the element a_p in the same place where the element a_l was, one needs to interchange the element a_p with the element next to it on the left k times. Together we need then $2k + 1$ interchanges of the neighbouring elements. It means that the class of the permutation will change odd number of times, and it will change from odd to even or from even to odd, if it was odd or even, respectively at the beginning.

Theorem 2. Both classes of permutations are equinumerous.

Every even permutation one can transform into an odd permutation, and every odd permutation one can transform into even permutation. Because there is $n!$ permutations of n elements, there must be $n!/2$ even permutations and $n!/2$ odd permutations of n elements.

References

- [1] Swietoslaw Romanowski and Włodzimierz Wrona (1967) *Matematyka wyższa dla studiów technicznych* Warszawa, Państwowe Wydawnictwo Naukowe

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