Proof that the sequence $(1+1/n)^n$ converges to the number e

We have the binomial theorem

$$(a+b)^N = \sum_{k=0}^{N} \binom{n}{k} a^k b^{n-k} \tag{1}$$

For the sequence $(1+1/n)^n$ we can write

$$\left(\frac{1}{n}+1\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \tag{2}$$

$$\left(\frac{1}{n}+1\right)^n = \binom{n}{0}\frac{1}{n^0} + \binom{n}{1}\frac{1}{n^1} + \binom{n}{2}\frac{1}{n^2} + \dots + \binom{n}{n}\frac{1}{n^n} \tag{3}$$

what gives

$$\begin{aligned}\left(\frac{1}{n}+1\right)^n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} \qquad (4)\\ +\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \dots &+ \frac{n(n-1)(n-2)\dots 1}{n!}\frac{1}{n^n}\\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\\ &+ \dots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{m-1}{n}\right)\\ &+ \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right)\end{aligned}$$

$$\begin{aligned}T(x) &= \binom{n}{x}\left(\frac{1}{n}\right)^x \qquad (5)\\ &= \frac{n!}{x!(n-x)!}\left(\frac{1}{n}\right)^x\end{aligned}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!\,n^x}$$
$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!}$$

Now as $n \to \infty$,

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) \to 1 \tag{6}$$

and

$$T(x) = \frac{1}{x!} \tag{7}$$

and

$$e = \sum_{x=0}^{\infty} T(x) \tag{8}$$

References

[1] Maor, E. (1994) *e The Story of a Number* Princeton University Press, Princeton, New Jersey, NJ

Pawel J Piskorz (paweljs@gmail.com)