Proof that the number $e > 2$ and $e < 3$

For $f(x) = e^x$ we have the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n \tag{1}$$

because all derivatives of e^x are just e^x. Substituting $a = 0$ in the Taylor series we receve the Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{2}$$

We have for $x = 1$

$$\begin{aligned} S_N = \sum_{n=0}^{N} \frac{1}{n!} &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{N!} \\ &< 1 + (\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N-1}}) \end{aligned} \tag{3}$$

We have the following formula for the sum of the geometric series

$$Z(N) = \sum_{n=0}^{N} x^n = 1 + x + x^2 + \ldots + x^N \tag{4}$$

$$Z(N) = \sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x} \tag{5}$$

In the inequality (3) we have the sum

$$Z(N-1) = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N-1}} \tag{6}$$

$$Z(N-1) = \frac{1 - x^N}{1 - x} \tag{7}$$

and the limit for $x = \frac{1}{2}$

$$\lim_{N\to\infty} Z(N-1) = \lim_{N\to\infty} \frac{1-x^N}{1-x} = \frac{1}{1-\frac{1}{2}} = 2 \tag{8}$$

Now we can write

$$\lim_{N\to\infty} S(N) = e \tag{9}$$

and we have

$$\lim_{N\to\infty} S(N) < 1 + \lim_{N\to\infty} Z(N-1) \tag{10}$$

$$e < 1 + 2 \tag{11}$$

what gives

$$e < 3 \tag{12}$$

From the equation (3) it also can be seen that the number e is greater than the first two terms $1+1$ of the $S(N)$ series and

$$2 < e < 3 \tag{13}$$

References

[1] Maor, E. (1994) *e The Story of a Number* Princeton University Press, Princeton, New Jersey, NJ

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