Jacobian for n-Dimensional Spherical Coordinates

In this article we will derive the general formula for the Jacobian of the transformation from the Cartesian coordinates to the spherical coordinates in n dimensions without the use of determinants.

In general, the equation for the sphere of radius R in integer n dimensions is

$$x_1^2 + x_2^2 + \ldots + x_n^2 = R^2 \tag{1}$$

where $x_1, x_2, \ldots, x_n$ are Cartesian coordinates. The n-dimesnsional sphere is often called n-hypersphere. For $n = 2$ we have just the equation of a circle, and for $n = 3$ the equation of a three-dimensional sphere. To compute the area of a circle or the volume of a three-dimensional sphere it is convenient to carry out the appropriate integrations in azimuthal and spherical coordinates, respectively. The computation of the volume of the n-dimensional sphere would require integration in n-dimensional spherical coordinates. The derivation of the transformation from the Cartesian coordinates $x_1, x_2, \ldots, x_n$ to the n-dimensional spherical coordinates $r, \theta, \phi_1, \ldots \phi_{n-2}$ has been presented in [1]. For example the transformation for five dimensions is given by equations (19) and for n dimensions is

$$\begin{aligned}
1:&\quad x_1 = r\cos\phi_1 \\
2:&\quad x_2 = r\sin\phi_1\cos\phi_2 \\
3:&\quad x_3 = r\sin\phi_1\sin\phi_2\cos\phi_3 \\
\ldots& \\
i:&\quad x_i = r\sin\phi_1\sin\phi_2\ldots\sin\phi_{i-1}\cos\phi_i \\
\ldots& \\
n-2:&\quad x_{n-2} = r\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-3}\cos\phi_{n-2} \\
n-1:&\quad x_{n-1} = r\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-2}\cos\theta \\
n:&\quad x_n = r\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-2}\sin\theta
\end{aligned} \tag{2}$$

where $0 \leq \phi_i \leq \pi, i = 1, \ldots, n-2$ and $0 \leq \theta \leq 2\pi$. The n-dimensional spherical coordinates are created in such way that they are orthogonal what means that the scalar product of their any two basis vectors, which are sometimes called versors, $\hat{i}_r, \hat{i}_\theta, \hat{i}_{\phi_i}$ for $i = 1, 2, \ldots, n-2$ is equal zero. The n-dimensional Cartesian coordinates are also orthogonal.

The transformation from one set of coordinates to another one involves the change of the infinitesimally small volume element. In Cartesian coordinates the volume element is simply

$$dV^{(x_1,x_2,\ldots,x_n)} = dx_1 dx_1 \ldots dx_n \tag{3}$$

and the change from the Cartesian to the spherical coordinates involves the Jacobian $J(r, \theta, \phi_1, \phi_2, \ldots, \phi_{n-2})$ of the transformation, so we must write the formula for the volume element in the n-dimensional spherical coordinates as

$$dV^{(r,\theta,\phi_1,\phi_2,\ldots,\phi_{n-2})} = J(r, \theta, \phi_1, \phi_2, \ldots, \phi_{n-2}) dr d\theta d\phi_1 d\phi_2 \ldots d\phi_{n-2} \tag{4}$$

The Jacobian is a determinant of the n by n matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial \phi_1} & \cdots & \frac{\partial x_1}{\partial \phi_{n-2}} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_2}{\partial \phi_1} & \cdots & \frac{\partial x_2}{\partial \phi_{n-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial r} & \frac{\partial x_n}{\partial \theta} & \frac{\partial x_n}{\partial \phi_1} & \cdots & \frac{\partial x_n}{\partial \phi_{n-2}} \end{pmatrix} \tag{5}$$

For how Jacobian determinant emerges in the transformation of variables we will point the reader to [2].

We will analyze the Jacobians of transformations from the Cartesian to the spherical coordinates for dimensions $n = 1, 2, 3, 4, 5$ without actually computing any determinants, and we will develop the general formula for the Jacobian of the transformation of coordinates for any dimension $n > 2$. Computing the Jacobian determinants even for a three-dimensional spherical coordinates transformation is cumbersome. We will employ another method which is based on the definition of the angle measure in radians and on the orthogonality of the spherical coordinates.

The radian measure $d\alpha$ of a central angle of a circle is defined as the ratio of the length dl_α of the arc the angle subtends divided by the radius r of the circle

$$d\alpha = \frac{dl_\alpha}{r} \tag{6}$$

We may express the value of the volume element $dV^{(r,\theta,\phi_1,\phi_2,\ldots,\phi_{n-2})}$ as

$$dV^{(r,\theta,\phi_1,\phi_2,\ldots,\phi_{n-2})} = dr dl_\theta dl_{\phi_1} dl_{\phi_2} \ldots dl_{\phi_{n-2}} \tag{7}$$

by virtue of orthogonality of the versors $\hat{i}_r, \hat{i}_\theta, \hat{i}_{\phi_i}$ for $i = 1, 2, \ldots, n-2$ along the radius r and tangent to the coordinate lines θ, ϕ_i for $i = 1, 2, \ldots, n-2$, respectively at the point $(r, \theta, \phi_1, \ldots, \phi_{n-2})$. The versors for threedimesional spherical coordinates which are denoted in this article by $\hat{i}_r, \hat{i}_\theta, \hat{i}_{\phi_1}$ are illustrated in [3]. θ is azimuthal angle coordinate, and ϕ_i is called i-th polar angle coordinate.

For $n = 1$

$$1: \quad x_1 = r \tag{8}$$

we just make a variable substitution and

$$dV^{(r)} = J(r)dr = dr \tag{9}$$

what gives $J_1 = J(r) = 1$.

For $n = 2$ we add azimuthal angle θ as the second coordinate

$$\begin{aligned} 1: \quad x_1 &= r\cos\theta \\ 2: \quad x_2 &= r\sin\theta \end{aligned} \tag{10}$$

and we have for

$$\theta: \quad d\theta = dl_\theta / r \tag{11}$$

The volume element for $n = 2$ is

$$dV^{(r,\theta)} = J(r,\theta)drd\theta = drdl_\theta = rdrd\theta \tag{12}$$

and $J_2 = J(r, \theta) = r$.

For $n = 3$ we need to add to the coordinates the polar angle ϕ_1

$$\begin{aligned} 1: \quad x_1 &= r\cos\phi_1 \\ 2: \quad x_2 &= r\sin\phi_1\cos\theta \\ 3: \quad x_3 &= r\sin\phi_1\sin\theta \end{aligned} \tag{13}$$

and we have for

$$\begin{aligned} \phi_1: \quad d\phi_1 &= dl_{\phi_1}/r \\ \theta: \quad d\theta &= dl_\theta/(r\sin\phi_1) \end{aligned} \tag{14}$$

We come to the above formulas just by taking into account that the angle $d\phi_1$ subtends the arc of length dl_{ϕ_1} of the radius r, and that the angle $d\theta$ subtends the arc of length dl_θ of the radius $r\sin\phi_1$. The volume element in 3 dimensions is

$$dV^{(r,\theta,\phi_1)} = J(r,\theta,\phi_1)drd\theta d\phi_1 = drdl_\theta dl_{\phi_1} = r^2\sin\phi_1 drd\theta d\phi_1 \tag{15}$$

and the Jacobian $J_3 = J(r,\theta,\phi_1) = r^2\sin\phi_1$.

For $n = 4$ we add to the coordinates the polar angle ϕ_2

$$\begin{aligned}
1:\quad & x_1 = r\cos\phi_1 \\
2:\quad & x_2 = r\sin\phi_1\cos\phi_2 \\
3:\quad & x_3 = r\sin\phi_1\sin\phi_2\cos\theta \\
4:\quad & x_4 = r\sin\phi_1\sin\phi_2\sin\theta
\end{aligned} \tag{16}$$

and it gives for

$$\begin{aligned}
\phi_1:\quad d\phi_1 &= dl_{\phi_1}/r \\
\phi_2:\quad d\phi_2 &= dl_{\phi_2}/(r\sin\phi_1) \\
\theta:\quad d\theta &= dl_\theta/(r\sin\phi_1\sin\phi_2)
\end{aligned} \tag{17}$$

Now the situation is that the angle $d\phi_1$ subtends the arc of length dl_{ϕ_1} of the radius r as before, the angle $d\phi_2$ subtends the arc of length dl_{ϕ_2} of the radius $r\sin\phi_1$, which is the new radius as given in (16) in the formula for x_2, and that the angle $d\theta$ subtends the arc of length dl_θ of the radius $r\sin\phi_1\sin\phi_2$ also by analogy to the situation for 3 dimensions. In other words we develop the above relations as a consequence of the definition of the spherical coordinates in 3 dimensions in equations (13) and by following the relations in (14). The volume element in 4 dimensions is

$$\begin{aligned}
dV^{(r,\theta,\phi_1,\phi_2)} &= J(r,\theta,\phi_1,\phi_2)drd\theta d\phi_1 d\phi_2 = drdl_\theta dl_{\phi_1}dl_{\phi_2} \\
&= r^3\sin^2\phi_1\sin\phi_2 drd\theta d\phi_1 d\phi_2
\end{aligned} \tag{18}$$

and the Jacobian $J_4 = J(r,\theta,\phi_1,\phi_2) = r^3\sin^2\phi_1\sin\phi_2$.

For $n = 5$ we have

$$\begin{aligned}
1:\quad & x_1 = r\cos\phi_1 \\
2:\quad & x_2 = r\sin\phi_1\cos\phi_2 \\
3:\quad & x_3 = r\sin\phi_1\sin\phi_2\cos\phi_3 \\
4:\quad & x_4 = r\sin\phi_1\sin\phi_2\sin\phi_3\cos\theta \\
5:\quad & x_5 = r\sin\phi_1\sin\phi_2\sin\phi_3\sin\theta
\end{aligned} \tag{19}$$

and for

$$\begin{array}{rcl} \phi_1: \ d\phi_1 &=& dl_{\phi_1}/r \\ \phi_2: \ d\phi_2 &=& dl_{\phi_2}/(r\sin\phi_1) \\ \phi_3: \ d\phi_3 &=& dl_{\phi_3}/(r\sin\phi_1\sin\phi_2) \\ \theta: \ d\theta &=& dl_\theta/(r\sin\phi_1\sin\phi_2\sin\phi_3) \end{array} \tag{20}$$

This gives the volume element for 5 dimensions

$$\begin{aligned} dV^{(r,\theta,\phi_1,\phi_2,\phi_3)} &= J(r,\theta,\phi_1,\phi_2,\phi_3)drd\theta d\phi_1 d\phi_2 d\phi_3 = drdl_\theta dl_{\phi_1}dl_{\phi_2}dl_{\phi_3} \\ &= r^4\sin^3\phi_1\sin^2\phi_2\sin\phi_3 drd\theta d\phi_1 d\phi_2 d\phi_3 \end{aligned} \tag{21}$$

and the Jacobian $J_5 = J(r,\theta,\phi_1,\phi_2,\phi_3) = r^4\sin^3\phi_1\sin^2\phi_2\sin\phi_3$. We notice that our Jacobian for 5 dimensions is just the product of the denominators from the equations (20).

The pattern for the Jacobian of the transformation from n Cartesian coordinate system to the system of n-dimensional spherical coordinates clearly reveals itself. For $n > 2$

$$J_n = J(r,\theta,\phi_1,\phi_2,\ldots,\phi_{n-2}) = r^{n-1}\prod_{k=1}^{n-2}\sin^{n-1-k}\phi_k \tag{22}$$

The Jacobian we derived may be used in computing the volume $V_n(c)$ or the surface $S_n(r)$ of a n-dimensional sphere of radius c or r, respectively.

$$\begin{aligned} V_n(c) &= \int_{r=0}^{c}\int_{\theta=0}^{2\pi}\int_{\phi_1=0}^{\pi}\int_{\phi_2=0}^{\pi}\cdots\int_{\phi_{n-2}=0}^{\pi} J_n\, drd\theta d\phi_1 d\phi_2\ldots d\phi_{n-2} \\ &= \int_{r=0}^{c} r^{n-1}\,dr\int_{\theta=0}^{2\pi}d\theta\prod_{k=1}^{n-2}\int_{\phi_k=0}^{\pi}\sin^{n-1-k}\phi_k\,d\phi_k \end{aligned} \tag{23}$$

$$\begin{aligned} S_n(r) &= \int_{\theta=0}^{2\pi}\int_{\phi_1=0}^{\pi}\int_{\phi_2=0}^{\pi}\cdots\int_{\phi_{n-2}=0}^{\pi} J_n\, d\theta d\phi_1 d\phi_2\ldots d\phi_{n-2} \\ &= r^{n-1}\int_{\theta=0}^{2\pi}d\theta\prod_{k=1}^{n-2}\int_{\phi_k=0}^{\pi}\sin^{n-1-k}\phi_k\,d\phi_k \end{aligned} \tag{24}$$

The further computation is an exercise in applying the formula for the integral of the type

$$\int_0^{\pi/2}\sin^n x\cos^m\,dx = \frac{1}{2}B\left(\frac{1}{2}(n+1),\frac{1}{2}(m+1)\right) \tag{25}$$

where B is the Beta function, which is defined in [4] and [5] to compute the integral of powers of sine, and then the application of the Euler gamma function Γ which is described in [4], [6] and [7] and which is related to the function Beta

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{26}$$

Properties of Euler Gamma function used in this article are presented also in [8]. For natural values of x the Euler Gamma function has the property

$$\Gamma(x) = (x-1)! \tag{27}$$

If we substitute $x+1$ for x in the above equation, then we obtain

$$\Gamma(x+1) = x! = x(x-1)! = x\Gamma(x) \tag{28}$$

The relation

$$\Gamma(x+1) = x\Gamma(x) \tag{29}$$

is valid also for real values of x. Also

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{30}$$

Then we have

$$S_n(r) = r^{n-1} 2\pi \frac{\Gamma^{n-2}(\frac{1}{2})}{\Gamma(\frac{1}{2}n)} = \frac{2\pi^{\frac{1}{2}n} r^{n-1}}{\Gamma(\frac{1}{2}n)} \tag{31}$$

$$V_n(R) = \int_{r=0}^{R} S_n(r)\,dr = \frac{2\pi^{\frac{1}{2}n} R^n}{n\Gamma(\frac{1}{2}n)} \tag{32}$$

In particular for $n=3$, i.e. for three dimensions we can obtain the formulas for the surface area and for the volume of a sphere.

For the sphere surface area from equation (31) we have

$$S_3(r) = \frac{2\pi^{\frac{3}{2}} r^2}{\Gamma(\frac{3}{2})} = \frac{2(\sqrt{\pi})^3 r^2}{\Gamma(\frac{3}{2})} \tag{33}$$

From equations (29) and (30) we can compute the value of $\Gamma\left(\frac{3}{2}\right)$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} \tag{34}$$

and substitute it into equation (33)

$$S_3(r) = \frac{2(\sqrt{\pi})^3 r^2}{\frac{1}{2}\sqrt{\pi}} \tag{35}$$

obtaining the well known formula for the surface area $S_3(r)$ of a three-dimensional sphere of radius r

$$S_3(r) = 4\pi r^2 \tag{36}$$

For the sphere volume $V_3(R)$ of a three-dimensional sphere from equation (32) we obtain

$$V_3(R) = \frac{2(\sqrt{\pi})^3 R^3}{3\Gamma(\frac{3}{2})} = \frac{2(\sqrt{\pi})^3 R^3}{\frac{3}{2}\sqrt{\pi}} \tag{37}$$

and we receive

$$V_3(R) = \frac{4}{3}\pi R^3 \tag{38}$$

what is also a familiar formula for the volume of a sphere of radius R.

References

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