Note on mathematical induction

Let us consider the series

$$S(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} \tag{1}$$

We notice that we have

$$\begin{aligned} S(1) &= \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} \\ S(2) &= \sum_{i=1}^{2} \frac{1}{i(i+1)} = \frac{1}{2} + \frac{1}{2(2+1)} \\ &= \frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\ S(3) &= \sum_{i=1}^{3} \frac{1}{i(i+1)} = \frac{2}{3} + \frac{1}{3(3+1)} \\ &= \frac{2}{3} + \frac{1}{12} = \frac{8}{12} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \\ S(4) &= \sum_{i=1}^{4} \frac{1}{i(i+1)} = \frac{3}{4} + \frac{1}{4(4+1)} \\ &= \frac{3}{4} + \frac{1}{20} = \frac{15}{20} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5} \end{aligned} \tag{2}$$

From equations (2) we may assume a hypothesis that in general

$$S(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \tag{3}$$

We will prove the above formula by mathematical induction if from the fact that based on the equation (3) for $S(n)$ we manage to show that

$$S(n+1) = \sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{n+2} \tag{4}$$

holds. To achieve that we compute

$$S(n+1) = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \tag{5}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \tag{6}$$

$$= \frac{n^2+2n+1}{(n+1)(n+2)}$$

The discriminant of the quadratic equation $n^2+2n+1=0$ is $\Delta = 4-4\cdot 1\cdot 1 = 0$ so this quadratic equation has a double root

$$n_{1,2} = \frac{-2}{2} = -1$$

what means that we can write

$$S(n+1) = \sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{(n+1)(n+1)}{(n+1)(n+2)} = \frac{n+1}{n+2} \tag{7}$$

The pattern holds if we transition from n to $n+1$ and the equation (3) is true.

It is interesting that the infinite series

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \lim_{n\to\infty} \frac{n}{n+1} \tag{8}$$

$$\lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \frac{1}{1+0} = 1$$

has the sum equal to 1.

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