# Note on intermediate computation for integral $\int_0^t d\tau \int_0^t d\tau' f(\tau - \tau')$

We want to compute the integral presented in [1] on page 420

$$I = \int_0^t d\tau \int_0^t d\tau' f(\tau - \tau') \tag{1}$$

Let us have the integral function of $f(\tau)$ as $F(\tau)$

$$\int d\tau f(\tau) = F(\tau) \tag{2}$$

and let us carry out the $\tau$ integration in $I$

$$I = \int_0^t d\tau' (F(t - \tau') - F(-\tau')) \tag{3}$$

We can always write

$$I = \int_0^t d\tau' F(t - \tau') - \int_0^t d\tau' F(-\tau') = \int_0^t d\tau' F(t - \tau') - \int_0^t du F(-u) \tag{4}$$

$$\int_0^t d\tau' F(t - \tau') = \left| \begin{array}{rcl} u & = & t - \tau' \\ du & = & -d\tau' \\ \tau' = 0 & \to & u = t \\ \tau' = t & \to & u = 0 \end{array} \right| = -\int_t^0 du F(u) = \int_0^t du F(u) \tag{5}$$

and we have

$$I = \int_0^t du F(u) - \int_0^t du F(-u) = \int_0^t du (F(u) - F(-u)) \tag{6}$$

Integrating by parts we have

$$\begin{aligned} I = \int_0^t du (F(u) - F(-u)) = \\ \left| \begin{array}{rcl} 1 & = & x' \\ F(u) - F(-u) & = & y \end{array} \right| \left| \begin{array}{rcl} x & = & u \\ y' & = & f(u) + f(-u) \end{array} \right| = \\ [u(F(u) - F(-u))]_{u=0}^{u=t} - \int_0^t du u (f(u) + f(-u)) \end{aligned} \tag{7}$$

We notice that

$$[u(F(u) - F(-u))]_{u=0}^{u=t} = t(F(t) - F(-t)) \tag{8}$$

On the other hand we have

$$\begin{aligned} t\int_0^t du(f(u) + f(-u)) = t[(F(u) - F(-u))]_{u=0}^{u=t} \\ = t(F(t) - F(-t)) - t(F(0) - F(0)) = t(F(t) - F(-t)) \end{aligned} \tag{9}$$

Therefore the integral $I$ can be written as

$$I = t\int_0^t du(f(u) + f(-u)) - \int_0^t duu(f(u) + f(-u)) \tag{10}$$

and

$$I = \int_0^t d\tau \int_0^t d\tau' f(\tau - \tau') = \int_0^t du(t - u)(f(u) + f(-u)) \tag{11}$$

Having this result we are able to solve the Exercise 21-37 in [2]. Using our notation this exercise requires proof that

$$\int_0^t d\tau' \int_0^t d\tau'' f(\tau'' - \tau') = 2t\int_0^t du(1 - \frac{u}{t})f(u) \tag{12}$$

This exercise requires the assumption that $f(\tau) = f(-\tau)$.

We solve the exercise as follows:

$$\begin{aligned} \int_0^t d\tau' \int_0^t d\tau'' f(\tau'' - \tau') = \int_0^t du(t - u)(f(u) + f(-u)) \\ = t\int_0^t du(f(u) + f(-u)) - \int_0^t duu(f(u) + f(-u)) \\ = 2t\int_0^t duf(u) - 2\int_0^t duuf(u) \\ = 2t\int_0^t du(1 - \frac{u}{t})f(u) \end{aligned} \tag{13}$$

# References


[1] Schwabl, F. (2006) *Statistische Mechanik* 3. Auflage, Springer, Berlin, Heidelberg, New York

[2] McQuarrie, D.A. (2000) *Statistical Mechanics* University Science Books, Sausalito California

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