Integral of quadratic form example

Let Y_n is n-dimensional vector

$$Y_n = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{1}$$

By primed symbol Y_n' we denote the transposition

$$Y_n' = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \tag{2}$$

and M_n is a positive definite $n \times n$ symmetric matrix independent of Y_n. We wish to evaluate the integral of

$$\phi_n(Y_n) = e^{-\frac{1}{2}Y_n' M_n^{-1} Y_n} \tag{3}$$

over all n-dimensional Euclidean space

$$v = \int_{-\infty}^{\infty} \phi_n(Y_n)\, dY_n \tag{4}$$

where $dY_n = dy_1 dy_2 \cdots dy_n$ is the element of volume.

Since M_n is positive definite, there exists a nonsingular square matrix Q_n such that

$$M_n = Q_n' Q_n \tag{5}$$

Then

$$M_n^{-1} = (Q_n' Q_n)^{-1} = Q_n^{-1} Q_n^{'-1} \tag{6}$$

and we can choose a transformation

$$Z_n = Q_n'^{\,-1} Y_n \tag{7}$$

so we have

$$-\frac{1}{2} Y_n' M_n^{-1} Y_n = -\frac{1}{2} Y_n' Q_n^{-1} Q_n^{'-1} Y_n = -\frac{1}{2} Z_n' Z_n \tag{8}$$

Based on equation (7) we have the Jacobian J such that we use JdZ_n instead of dY_n with the new function of new quadratic form

$$e^{-\frac{1}{2}Z_n' Z_n} \tag{9}$$

and

$$J = \left| \frac{\partial(Y_n)}{\partial(Z_n)} \right| \tag{10}$$

Because we have a linear transformation

$$Y_n = Q'_n Z_n \tag{11}$$

then the Jacobian J is equal to the determinant $|Q'_n|$ of matrix Q'_n which is equal to the determinant $|Q_n|$ of matrix Q_n

$$J = |Q_n| \tag{12}$$

Because of equation (5) and because the determinant of the product of the given matrices equals to the product of determinants of the given matrices we have

$$J = |Q_n| = \sqrt{|M_n|} \tag{13}$$

Then the integral

$$\begin{aligned} v = \int_{-\infty}^{\infty} e^{-\frac{1}{2}Y'_n M_n^{-1} Y_n} \, dY_n = \int_{-\infty}^{\infty} J e^{-\frac{1}{2}Z'_n Z_n} \, dZ_n \\ = \sqrt{|M_n|} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \, dz \right)^n = \big(2\pi\big)^{\frac{1}{2}n} \sqrt{|M_n|} \end{aligned} \tag{14}$$

We have then the normalization factor $\big(2\pi\big)^{\frac{1}{2}n} \sqrt{|M_n|}$ such that

$$f_n(Y_n) = \frac{1}{\big(2\pi\big)^{\frac{1}{2}n} \sqrt{|M_n|}} e^{-\frac{1}{2}Y'_n M_n^{-1} Y_n} \, dY_n \tag{15}$$

and

$$\int_{-\infty}^{\infty} f_n(Y_n) \, dY_n = 1 \tag{16}$$

Now we will compute

$$\begin{aligned} (Y_n - A_n)' M_n^{-1} (Y_n - A_n) = Y'_n M_n^{-1} Y_n - Y'_n M_n^{-1} A_n \\ - A'_n M_n^{-1} Y_n + A'_n M_n^{-1} A_n \end{aligned} \tag{17}$$

where A_n is a vector. Let us substitute $A_n = M_n T_n$ where T_n is a vector. Then we have

$$(Y_n - M_n T_n)' M_n^{-1} (Y_n - M_n T_n) = Y'_n M_n^{-1} Y_n - Y'_n T_n - T'_n Y_n + T'_n M_n T_n \tag{18}$$

The equation (18) can be rewritten as

$$(Y_n - M_nT_n)'M_n^{-1}(Y_n - M_nT_n) = Y_n'M_n^{-1}Y_n - 2T_n'Y_n + T_n'M_nT_n \tag{19}$$

and

$$Y_n'M_n^{-1}Y_n - 2T_n'Y_n = (Y_n - M_nT_n)'M_n^{-1}(Y_n - M_nT_n) - T_n'M_nT_n \tag{20}$$

Multiplying the abve equation by $-\frac{1}{2}$ we receive

$$T_n'Y_n - \frac{1}{2}Y_n'M_n^{-1}Y_n = -\frac{1}{2}(Y_n - M_nT_n)'M_n^{-1}(Y_n - M_nT_n) + \frac{1}{2}T_n'M_nT_n \tag{21}$$

Now let us compute

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{T_n'Y_n} e^{-\frac{1}{2}Y_n'M_n^{-1}Y_n}\, dY_n \\ = e^{\frac{1}{2}T_n'M_nT_n} &\int_{-\infty}^{\infty} e^{-\frac{1}{2}(Y_n-M_nT_n)'M_n^{-1}(Y_n-M_nT_n)}\, dY_n \\ &= \left(2\pi\right)^{\frac{1}{2}n}\sqrt{|M_n|}\, e^{\frac{1}{2}T_n'M_nT_n} \end{aligned} \tag{22}$$

In the above calculation we have used the result from the equation (14) however for a shifted expnonent variable.

We have then a number $m_n(T_n)$ depending on n components of the vector T_n from integral

$$m_n(T_n) = \frac{1}{\left(2\pi\right)^{\frac{1}{2}n}\sqrt{|M_n|}} \int_{-\infty}^{\infty} e^{T_n'Y_n} e^{-\frac{1}{2}Y_n'M_n^{-1}Y_n}\, dY_n = e^{\frac{1}{2}T_n'M_nT_n} \tag{23}$$

Now let us compute

$$\frac{\partial}{\partial t_i} m_n(T_n)\Big|_{T_n=O_n} \qquad \text{and} \qquad \frac{\partial^2}{\partial t_i t_j} m_n(T_n)\Big|_{T_n=O_n} \tag{24}$$

where O_n is a vector having n zeros as its components

$$O_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{25}$$

We have then

$$\frac{\partial}{\partial t_i} m_n(T_n)\Big|_{T_n=O_n} = \int_{-\infty}^{\infty} y_i f_n(Y_n)\, dY_n \tag{26}$$

and

$$\frac{\partial^2}{\partial t_i t_j} m_n(T_n)\Big|_{T_n=O_n} = \int_{-\infty}^{\infty} y_i y_j f_n(Y_n)\, dY_n \tag{27}$$

Taking the appropriate derivatives of the right hand side of the equation (23) we receive

$$\begin{aligned} \frac{\partial}{\partial t_i} m_n(T_n)\Big|_{T_n=O_n} &= \frac{\partial}{\partial t_i} e^{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j m_{i,j}}\Big|_{T_n=O_n} \\ &= \sum_{j=1}^{n} m_{i,j} t_j e^{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j m_{i,j}}\Big|_{t_i=t_j=0} = 0 \end{aligned} \tag{28}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial t_i t_j} m_n(T_n)\Big|_{T_n=O_n} &= \frac{\partial}{\partial t_j} \sum_{j=1}^{n} m_{i,j} t_j e^{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j m_{i,j}}\Big|_{t_i=t_j=0} \\ &= m_{i,j} e^{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j m_{i,j}}\Big|_{t_i=t_j=0} \\ &+ \sum_{j=1}^{n} \left[m_{i,j} t_j \left(\sum_{i=1}^{n} m_{i,j} t_i e^{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j m_{i,j}} \right) \right]\Big|_{t_i=t_j=0} = m_{i,j} \end{aligned} \tag{29}$$

We have as the result

$$\frac{\partial}{\partial t_i} m_n(T_n)\Big|_{T_n=O_n} = \int_{-\infty}^{\infty} y_i f_n(Y_n)\, dY_n = 0 \tag{30}$$

and

$$\frac{\partial^2}{\partial t_i t_j} m_n(T_n)\Big|_{T_n=O_n} = \int_{-\infty}^{\infty} y_i y_j f_n(Y_n)\, dY_n = m_{i,j} \tag{31}$$

This is a very important result which we may rewrite as

$$\frac{\partial}{\partial t_i} m_n(T_n)\Big|_{T_n=O_n} = \frac{1}{(2\pi)^{\frac{1}{2}n}\sqrt{|M_n|}} \int_{-\infty}^{\infty} y_i e^{-\frac{1}{2}Y_n' M_n^{-1} Y_n}\, dY_n = 0 \tag{32}$$

and

$$\frac{\partial^2}{\partial t_i t_j} m_n(T_n)\Big|_{T_n=O_n} = \frac{1}{(2\pi)^{\frac{1}{2}n}\sqrt{|M_n|}} \int_{-\infty}^{\infty} y_i y_j e^{-\frac{1}{2}Y_n' M_n^{-1} Y_n}\, dY_n = m_{i,j} \tag{33}$$

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