Note on integration on a circle

Let us compute integral $\int_{C_r} (z - z_0)^n \, dz$, where n is an integer and C_r is the circle $|z - z_0| = r$ traversed once in the counterclockwise direction

$$\oint_{C_r} f(z) \, dz \tag{1}$$

A suitable parametrization for C_r is given by $z(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$. We can check that

$$z(t) - z_0 = re^{it} \tag{2}$$
$$(z(t) - z_0)(z(t) - z_0)^* = r^2 e^{it} e^{-it} \tag{3}$$
$$|z(t) - z_0|^2 = r^2 \tag{4}$$
$$|z(t) - z_0| = r \tag{5}$$

The * denotes a complex conjugate and we can use the formula for the circle

$$z(t) = z_0 + re^{it} \tag{6}$$

and we receive

$$f(z) = (z - z_0)^n = (z_0 + re^{it} - z_0)^n = r^n e^{int} \tag{7}$$

and

$$z'(t) = ire^{it} \tag{8}$$

Using the formula

$$\int_\gamma f(z) \, dz = \int_a^b f(z(t)) z'(t) \, dt \tag{9}$$

because

$$\frac{dz}{dt} = z'(t) \tag{10}$$
$$dz = z'(t) dt \tag{11}$$

we can write

$$\int_{C_r} (z - z_0)^n \, dz = \int_0^{2\pi} (r^n e^{int})(ire^{it}) \, dt \tag{12}$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt = ir^{n+1} \left. \frac{e^{i(n+1)t}}{i(n+1)} \right|_0^{2\pi}$$

$$= ir^{n+1} \left[\frac{1}{i(n+1)} - \frac{1}{i(n+1)} \right] = 0$$

If $n = -1$ then

$$\int_{C_r} (z - z_0)^{-1}\, dz = i r^{n+1} \int_0^{2\pi} dt = 2\pi i \tag{13}$$

$$\int_{C_r} (z - z_0)^n\, dz = \begin{cases} 0, & \text{if } n \neq -1 \\ 2\pi i, & \text{if } n = -1 \end{cases} \tag{14}$$

regardless of the value of r.

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