**The Simple Harmonic Oscillator Solved without Guessing**

We will consider the application of the Newton second law when the force acting on a material point is proportional to the displacement from the equilibrium point and directed to the equilibrium point [1]. As an example let us consider a one dimensional motion along the $x$ axis. The Newton equation then is

$$m\ddot{x} = -kx \tag{1}$$

where $x$ is displacement, $m$ is mass, $k$ is proportionality constant, and

$$\ddot{x} = \frac{d^2x}{dt^2} \tag{2}$$

is the second time $t$ derivative of the displacement, i.e. the accelleration. Quite similar equation we get when treating the long pendulum in small displacement angle approximation

$$m\ddot{x} = -mg(x/l) \tag{3}$$

what gives

$$l\ddot{x} = -gx \tag{4}$$

where $l$ is the length of the pendulum, $g$ is the acceleration due to gravity.

Our goal will be to solve the equation

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{5}$$

without making any assumptions about the solution function $x$ as suggested in [2]. We will make assumptions only about the initial conditions. The velocity $v$ of the motion is

$$\dot{x} = v = \frac{dx}{dt} \tag{6}$$

We start with noticing that

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

$$\frac{d(v^2)}{dx} = 2v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx} = \frac{d^2x}{dt^2}$$

The original equation (5) becomes

$$m\frac{1}{2}\frac{d(v^2)}{dx} + kx = 0 \tag{7}$$

$$d(v^2) + \frac{k}{m}2xdx = 0 \tag{8}$$

After integration

$$v^2 + \frac{k}{m}x^2 + C_1 = 0 \tag{9}$$

where $C_1$ is a constant. For maximum displacement $x = x_0$ the velocity $v = 0$ and

$$\frac{k}{m}x_0^2 + C_1 = 0 \tag{10}$$

$$C_1 = -\frac{k}{m}x_0^2 \tag{11}$$

what gives

$$v^2 = \frac{k}{m}\left(x_0^2 - x^2\right) \tag{12}$$

$$\frac{dx}{dt} = \sqrt{\frac{k}{m}}x_0\sqrt{1 - \left(\frac{x}{x_0}\right)^2} \tag{13}$$

Substituting $u = x/x_0$

$$\sqrt{\frac{k}{m}}\,dt = \frac{dx}{x_0\sqrt{1-(x/x_0)^2}} = \frac{du}{\sqrt{1-u^2}} \tag{14}$$

Integrating the equation (14) we receive

$$\arcsin u = \arcsin(x/x_0) = \sqrt{k/m}\,t + C_2 \tag{15}$$

and from that

$$x = x_0\sin(\sqrt{k/m}\,t + C_2) \tag{16}$$

We may assume the that the constant $C_2 = 0$ if for $t = 0$ the displacement $x = 0$, and our solution of the equation (5) is

$$x = x_0\sin(\sqrt{k/m}\,t) \tag{17}$$

The solution of equation (5) is a sinusoidal function. Sinusoidal function also describes the vibration of a string e.g. of a violin or a harp string. Such vibration corresponds to a harmonic tone and therefrom comes the name harmonic oscillation. Because of its solution (17) the equation (5) is called the equation of a harmonic oscillator.

Essentially, every point on a vibrating string vibrates like a harmonic oscillator. From the periodicity of the sinusoidal function we see then that the motion of the harmonic oscillator is periodic. Let us denote the period of the motion of the harmonic oscillator by $T$. Then

$$\begin{aligned} x(t) = x_0 \sin(\sqrt{k/m}\, t) = x(t) = x_0 \sin(\sqrt{k/m}\, t + 2\pi) \\ = x_0 \sin(\sqrt{k/m}\,(t+T)) = x_0 \sin(\sqrt{k/m}\, t + \sqrt{k/m}\, T) \end{aligned} \tag{18}$$

and from that

$$T\sqrt{k/m} = 2\pi \tag{19}$$

with the period $T$

$$T = 2\pi\sqrt{m/k} \tag{20}$$

We notice that for the pendulum we have the period

$$T = 2\pi\sqrt{l/g} \tag{21}$$

Frequency $\nu$ of the periodic motion denotes the number of periods per unit time

$$\nu = 1/T \tag{22}$$

For the harmonic oscillator

$$\nu = \frac{1}{2\pi}\sqrt{k/m} \tag{23}$$

and for the pendulum in the approximation of harmonic oscillations

$$\nu = \frac{1}{2\pi}\sqrt{g/l} \tag{24}$$

So our simple harmonic oscillator problem is solved. We had to know that

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C \tag{25}$$

where $C$ is constant. Let us prove the above equation (25) by asking how to compute the derivative of the arcsin function.

Let

$$y = \sin x \tag{26}$$

$$\frac{dy}{dx} = \frac{d\sin x}{dx} = \cos x = \sqrt{1-\sin^2 x} \tag{27}$$

With $y = \sin x$ and $x = \arcsin y$ we use the formula to calculate the derivative of the inverse function

$$\frac{dx}{dy} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{d\arcsin y}{dy} = \frac{1}{\sqrt{1-y^2}} \tag{28}$$

what is equivalent to

$$\frac{d\arcsin u}{du} = \frac{1}{\sqrt{1-u^2}} \tag{29}$$

and the equation (25) can be proven after integrating (29)

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \int d\ \arcsin u = \arcsin u + C \tag{30}$$

# References


[1] Guminski, K. (1964) *Elementy chemii teoretycznej (Elements of Theoretical Chemistry)* Panstwowe Wydawnictwo Naukowe (State Scientific Publishing), Warszawa

[2] Romanowski, S., Wrona, W. (1967) *Matematyka wyzsza dla studiow technicznych (Higher Mathematics for Technical Studies)* Panstwowe Wydawnictwo Naukowe (State Scientific Publishing), Warszawa



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