Note on geometric series

Partial sum with N terms of the geometric series is

$$S(N) = \sum_{n=0}^{N} x^n = 1 + x + x^2 + \ldots + x^N \tag{1}$$

If we multiply both sides of equation (1) by x, we receive

$$xS(N) = \sum_{n=0}^{N} x^{n+1} = x + x^2 + x^3 + \ldots + x^N + x^{N+1} \tag{2}$$

Now let us subtract the equation (2) from equation (1)

$$\begin{aligned} S(N) - xS(N) &= 1 - x^{N+1} \\ S(N)(1-x) &= 1 - x^{N+1} \\ S(N) &= \frac{1 - x^{N+1}}{1-x} \end{aligned} \tag{3}$$

We see that if $N \to \infty$ and $|x| < 1$

$$\lim_{N\to\infty} S(N) = \lim_{N\to\infty} \frac{1 - x^{N+1}}{1-x} = \frac{1}{1-x} \tag{4}$$

Then if $|x| < 1$, the series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \tag{5}$$

converges and the expansion is possible.

If the expansion in infinite power series is possible there should be a possibility to obtain it as the Maclaurin series expansion. Let us have

$$f(x) = \frac{1}{1-x} \tag{6}$$

Then

$$\begin{aligned} f^{(0)}(x) &= (1-x)^{-1} \\ f^{(1)}(x) &= ((1-x)^{-1})' = (1-x)^{-2} \\ f^{(2)}(x) &= ((1-x)^{-2})' = 2(1-x)^{-3} \\ f^{(3)}(x) &= (2(1-x)^{-3})' = 2 \cdot 3(1-x)^{-4} \\ f^{(4)}(x) &= (2 \cdot 3(1-x)^{-4})' = 2 \cdot 3 \cdot 4(1-x)^{-5} \\ &\ldots \\ f^{(n)}(x) &= n!(1-x)^{-(n+1)} \end{aligned} \tag{7}$$

The expansion in Maclaurin series is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{8}$$

with $a = 0$. We see that for $f(x) = 1/(1-x)$ we will receive

$$f^{(n)}(a) = n!(1-a)^{-(n+1)} \tag{9}$$

Substituting $a = 0$

$$f^{(n)}(a=0) = n!(1-0)^{-(n+1)} = n! \tag{10}$$

and then

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a=0)}{n!} x^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \tag{11}$$

and we have to remeber that this expansion is valid only if it is possible and it is possible when $|x| < 1$.

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