Notes on Fourier Transform Derivation from Fourier Series of Exponential Functions with Imaginary Exponents

It is possible to expand periodic function $f_T(x)$ having period T into Fourier series of exponential functions with imaginary exponents

$$f_T(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} \tag{1}$$

for $n = 0, \pm 1, \pm 2, \dots$ with

$$\omega_n = \frac{2\pi n}{T} \tag{2}$$

The coefficients c_n are given by the formula

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(x) e^{-i\omega_n x} \, dx \tag{3}$$

This formula is calculated as follows. The equation (1) is being multiplied by $e^{-i\omega_m x}$ and integrated in from $-T/2$ to $T/2$

$$\begin{aligned} \int_{-T/2}^{T/2} f_T(x) e^{-i\omega_m x} \, dx &= \sum_{n=-\infty}^{\infty} c_n \int_{-T/2}^{T/2} e^{i(\omega_n - \omega_m)x} \, dx \\ &= \sum_{n=-\infty}^{\infty} c_n \int_{-T/2}^{T/2} e^{i(n-m)\frac{2\pi}{T}x} \, dx \end{aligned} \tag{4}$$

The last integral gives

$$\int_{-T/2}^{T/2} e^{i(n-m)\frac{2\pi}{T}x} \, dx = [x]_{-T/2}^{T/2} = T \tag{5}$$

for $n = m$ and

$$\int_{-T/2}^{T/2} e^{i(n-m)\frac{2\pi}{T}x} \, dx = \frac{T}{2(n-m)\pi i} [e^{i(n-m)\frac{2\pi}{T}x}]_{-T/2}^{T/2} = 0 \tag{6}$$

for $n \neq m$.

Substituting in equation (1) the expression for the coefficients c_n we receive

$$f_T(x) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \Big(\int_{-T/2}^{T/2} f_T(\xi) e^{-i\omega_n \xi} \, d\xi \Big) e^{i\omega_n x} \tag{7}$$

Now we wish to consider the limit of the above expression as $T \to \infty$. We have from the definition of ω_n in equation (2)

$$\Delta\omega_n = \omega_{n+1} - \omega_n = \frac{2\pi(n+1)}{T} - \frac{2\pi n}{T} = \frac{2\pi}{T} \tag{8}$$

We notice that in the sum from equation (7)

$$f_T(x) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \left(\int_{-T/2}^{T/2} f_T(\xi) e^{-i\omega_n \xi} \, d\xi \right) e^{i\omega_n x} \Delta n \tag{9}$$

where $\Delta n = (n+1) - n = 1$, and we may use the following substitiution

$$\Delta n = \Delta\omega_n \frac{T}{2\pi} \tag{10}$$

changing the sum from sum over n to the sum over ω_n

$$f_T(x) = \frac{1}{2\pi} \sum_{\omega_n=-\infty}^{\infty} \left(\int_{-T/2}^{T/2} f_T(\xi) e^{-i\omega_n \xi} \, d\xi \right) e^{i\omega_n x} \Delta\omega_n \tag{11}$$

We have

$$\lim_{T\to\infty} = \Delta\omega_n = d\omega \tag{12}$$

Then it follows from the above that if we let $T \to \infty$ then the discrete variable ω_n approaches a continuous variable ω and the function $f_T(\xi)$ approaches the function $f(\xi)$ and the infinite sum becomes an integral from $-\infty$ to $+\infty$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} \, d\xi \right) e^{i\omega x} \, d\omega \tag{13}$$

The above equation is called a Fourier integral. We may rewrite this equation as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} \, d\xi \right) e^{i\omega x} \, d\omega \tag{14}$$

The integral

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} \, d\xi \tag{15}$$

is called a Fourier transform of the function $f(x)$. The function $f(x)$ may be computed from its Fourier transform as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \, d\omega \tag{16}$$

and the above integral is called an inverse Fourier transform of $F(\omega)$.

In some mathematical literature the pair of Fourier transforms is defined as

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} \, d\xi \tag{17}$$

and

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \, d\omega \tag{18}$$

but the definition

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} \, d\xi \tag{19}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \, d\omega \tag{20}$$

is preferred because of symmetry in the formulas for the simple and inverse transform.

Traditionally in the technical literature authors are using $\omega = 2\pi/T$ for the frequency domain and t for the time domain. Then we have the pair of Fourier transforms

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \tag{21}$$

for the simple Fourier transform and

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega \tag{22}$$

for the inverse Fourier transform. In the literature we find also the pair of Fourier transforms having $k = \frac{2\pi}{\lambda}$ for the wave vector domain where λ is the wave length and having distance x for the space domain

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \tag{23}$$

for the simple Fourier transform and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} \, dk \tag{24}$$

for the inverse Fourier transform.

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