Differentiation of Logarithmic Function

Let

$$y = \log_a(x) \tag{1}$$

$$\Delta y = \log_a(x + \Delta x) - \log_a(x) = \log_a \frac{x + \Delta x}{x} = \log_a \left(1 + \frac{\Delta x}{x}\right) \tag{2}$$

$$\begin{aligned} \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right) = \frac{1}{x}\frac{x}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right) \\ = \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x} \lim_{\Delta x \to 0} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x} \\ &= \frac{1}{x} \log_a \left[\lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right] \\ &= \frac{1}{x} \log_a e \end{aligned} \tag{4}$$

When $a = e$, $\log_a e = \log_e e = 1$, and $\log_e x = \ln x$ and we can write

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \tag{5}$$