Determinant 2 by 2

Let us consider the system of two linear equations

$$a_{1,1}x_1 + a_{1,2}x_2 = b_1 \tag{1}$$
$$a_{2,1}x_1 + a_{2,2}x_2 = b_2 \tag{2}$$

Let us multiply the equation (1) by $a_{2,2}$ and the equation (2) by $a_{1,2}$ what gives

$$a_{1,1}a_{2,2}x_1 + a_{1,2}a_{2,2}x_2 = b_1a_{2,2} \tag{3}$$
$$a_{1,2}a_{2,1}x_1 + a_{1,2}a_{2,2}x_2 = b_2a_{1,2} \tag{4}$$

Now we subtract equation (4) from equation (3) and we receive

$$(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})x_1 = b_1a_{2,2} - b_2a_{1,2} \tag{5}$$

Let us multiplay equation (1) by $a_{2,1}$ and equation (2) by $a_{1,1}$ to receive

$$a_{1,1}a_{2,1}x_1 + a_{1,2}a_{2,1}x_2 = b_1a_{2,1} \tag{6}$$
$$a_{1,1}a_{2,1}x_1 + a_{1,1}a_{2,2}x_2 = b_2a_{1,1} \tag{7}$$

and we subtract equation (6) from equation (7)

$$(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})x_2 = b_2a_{1,1} - b_1a_{2,1} \tag{8}$$

Equations (5) and (8) can be rewritten as follows

$$\det A = a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \tag{9}$$

$$\det A_{x_1} = b_1a_{2,2} - b_2a_{1,2} = \begin{vmatrix} b_1 & a_{1,2} \\ b_2 & a_{2,2} \end{vmatrix} \tag{10}$$

$$\det A_{x_2} = b_2 a_{1,1} - b_1 a_{2,1} = \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix} \tag{11}$$

where $\det A$ is determinant of the system of linear equations (1) and (2). To compute the x_1 and x_2 the determinants $\det A_{x_1}$ and $\det A_{x_2}$ are needed. Then we have

$$x_1 = \frac{\det A_{x_1}}{\det A} = \frac{\begin{vmatrix} b_1 & a_{1,2} \\ b_2 & a_{2,2} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} \tag{12}$$

and

$$x_2 = \frac{\det A_{x_2}}{\det A} = \frac{\begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} \tag{13}$$

The way of computing the x_1 and x_2 using the equations (12) and (13) is called Cramer's rule. We see that if $\det A = 0$ the system of linear equations (1) and (2) has no solutions.

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