Complex integration example

Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx, \quad 0 < a < 1 \tag{1}$$

Consider the integral

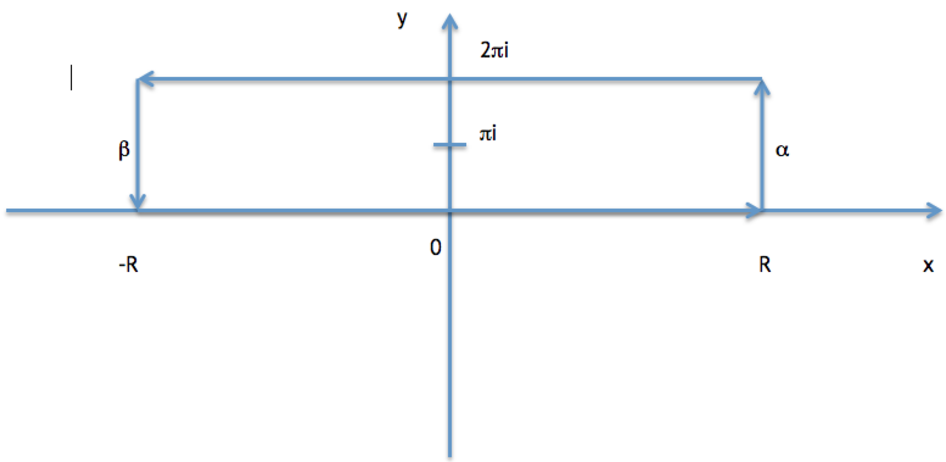

Figure 1: Plot of integration path enclosing the residue at $(0, \pi i)$

$$\int_{C} \frac{e^{az}}{1+e^{z}}\,dz \tag{2}$$

The function

$$\frac{e^{az}}{1+e^{z}} \tag{3}$$

has an infinite number of singularities in both the upper and the lower half-planes at points $z = \pi i(2n+1)$ where $n = 0, \pm 1, \pm 2, \ldots$ Of these only $(0, \pi i)$ lies inside contour C.

We compute the residue at πi as follows

$$Res(\pi i) = \left.\frac{e^{az}}{\frac{d}{dz}\Big(1+e^{z}\Big)}\right|_{z=\pi i} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i} \tag{4}$$

and we have

$$\int_C \frac{e^{az}}{1+e^z}\,dz = -2\pi i e^{a\pi i} \tag{5}$$

But

$$\begin{aligned}\int_C \frac{e^{az}}{1+e^z}\,dz &= \int_{-R}^{R}\left[\frac{e^{ax}}{1+e^x} - \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\right]dx \\ &\quad + \int_\alpha \frac{e^{az}}{1+e^z}\,dz + \int_\beta \frac{e^{az}}{1+e^z}\,dz\end{aligned} \tag{6}$$

Integration along the path α is when $z = R + iy$ with $0 \le y \le 2\pi$

$$\left|\frac{e^{az}}{1+e^z}\right| \le \frac{|e^{az}|}{|e^z|-1} = \frac{e^{aR}}{e^R-1} \tag{7}$$

Hence

$$\left|\int_\alpha \frac{e^{az}}{1+e^z}\,dz\right| \le \frac{2\pi e^{aR}}{e^R-1} \tag{8}$$

$$\lim_{R\to\infty} 2\pi\frac{e^{aR}}{e^R-1} = 2\pi \lim_{R\to\infty}\frac{e^{-R(1-a)}}{1-e^{-R}} = 0 \tag{9}$$

Similarly, on β $z = -R + i(2\pi - t), 0 \le t \le 2\pi$

$$\begin{aligned}\left|\int_\beta \frac{e^{az}}{1+e^z}\,dz\right| &= \left|\int_0^{2\pi} \frac{e^{a[-R+i(2\pi-t)]}}{1+e^{-R+i(2\pi-t)]}}(-i)\,dt\right| \\ &\le 2\pi\frac{e^{-aR}}{1-e^{-R}}\end{aligned} \tag{10}$$

$$\lim_{R\to\infty} 2\pi\frac{e^{-aR}}{1-e^{-R}} = 0 \tag{11}$$

$$\begin{aligned}\int_C \frac{e^{az}}{1+e^z}\,dz &= \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{ax}}\,dx - e^{a2\pi i}\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{ax}}\,dx \\ &= (1-e^{a2\pi i})\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{ax}}\,dx = -2\pi i e^{a\pi i}\end{aligned} \tag{12}$$

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{ax}}\,dx = \frac{2\pi i e^{a\pi i}e^{-a\pi i}}{e^{-a\pi i}(e^{a2\pi i}-1)} = \frac{2\pi i}{e^{a\pi i}-e^{-a\pi i}} = \frac{2\pi i}{2i\sin a\pi} = \frac{\pi}{\sin a\pi} \tag{13}$$

References

[1] Mitrinovic, D.S., Michael, J.H. (1966) *Calculus of Residues* P. Nordhoff LTD, Groningen, The Netherlands, page 58.

[2] Saff, E.B., Snider, A.D. (2015) *Fundamentals of Complex Analysis* Pearson, page 324.

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