

# Spherically symmetrical probability density functions and the chi-squared distribution

Let us have a spherically symmetrical probability density function

$$f_{\mathbf{x}}(x_1, x_2, \dots, x_n) = p(\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}) \quad (1)$$

everywhere in  $\mathbb{R}^n$ , where  $r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$  and  $p(r)$  is a nonnegative function of the single variable  $r$ . To subject this joint density function to the normalization condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_{\mathbf{x}}(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n = 1 \quad (2)$$

we must evaluate its integral over all of  $\mathbb{R}^n$ . We can do so by utilizing the fact that the probability density function  $f_{\mathbf{x}}(x_1, x_2, \dots, x_n)$  is constant over a hyperspherical shell. The volume  $dV$  of a hyperspherical shell of radius  $r$  and thickness  $dr$  in  $n$ -dimensional space is proportional to  $r^{n-1} dr$

$$dV = k_n r^{n-1} dr \quad (3)$$

Hence

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_{\mathbf{x}}(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n = k_n \int_0^{\infty} r^{n-1} p(r) dr \quad (4)$$

= 1

The constant  $k_n$  can be worked out by picking a particular joint density function for which both sides of equation (4) can be easily evaluated. A good function for that evaluation is a Gaussian function

$$f_{\mathbf{x}}(x_1, x_2, \dots, x_n) = (2\pi)^{-n/2} e^{-(x_1^2 + x_2^2 + \dots + x_n^2)/2} \quad (5)$$

for which

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-(x_1^2 + x_2^2 + \dots + x_n^2)/2} dx_1 dx_2 \dots dx_n \quad (6) \\ &= \int_{-\infty}^{\infty} e^{-x_1^2/2} dx_1 \int_{-\infty}^{\infty} e^{-x_2^2/2} dx_2 \dots \int_{-\infty}^{\infty} e^{-x_n^2/2} dx_n = (2\pi)^{n/2} \end{aligned}$$

and after changing the variables of integration

$$k_n \int_0^\infty r^{n-1} e^{-r^2/2} dr = (2\pi)^{n/2} \quad (7)$$

and if

$$f_{\mathbf{x}}(x_1, x_2, \dots, x_n) = (2\pi)^{-n/2} e^{-(x_1^2 + x_2^2 + \dots + x_n^2)/2} \quad (8)$$

then

$$k_n (2\pi)^{-n/2} \int_0^\infty r^{n-1} e^{-r^2/2} dr = 1 \quad (9)$$

We now compute the integral in equation (9)

$$\begin{aligned} \int_0^\infty r^{n-1} e^{-r^2/2} dr &= \left| \begin{array}{l} r^2/2 = u \\ r dr = du \\ r = (2u)^{1/2} \end{array} \right| = \int_0^\infty r^{n-2} e^{-r^2/2} r dr \\ &= \int_0^\infty (2u)^{(n-2)/2} e^{-u} du = \frac{1}{2} 2^{n/2} \int_0^\infty u^{n/2-1} e^{-u} du = \frac{1}{2} 2^{n/2} \Gamma(n/2) \end{aligned} \quad (10)$$

and we have

$$k_n (2\pi)^{-n/2} \frac{1}{2} 2^{n/2} \Gamma(n/2) = 1 \quad (11)$$

what gives

$$k_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad (12)$$

and

$$f_{\mathbf{x}} dx = p(x) k_n x^{n-1} dx \quad (13)$$

$$f_{\mathbf{x}} = \frac{2\pi^{n/2} x^{n-1} p(x)}{\Gamma(n/2)} \quad (14)$$

what we can rewrite as

$$f_{\mathbf{z}} = 2^{-(n/2)+1} [\Gamma(n/2)]^{-1} z^{n-1} e^{-z^2/2} \quad (15)$$

for

$$\mathbf{z} = (\mathbf{x}_1^2 + \mathbf{x}_2^2 + \dots + \mathbf{x}_n^2)^{1/2} \quad (16)$$

Now we introduce the variable

$$\mathbf{w} = \mathbf{z}^2 = \mathbf{x}_1^2 + \mathbf{x}_2^2 + \dots + \mathbf{x}_n^2 \quad (17)$$

and the question is what is the probability distribution function  $f_{\mathbf{w}}(\mathbf{w})$  if we know the function  $f_{\mathbf{z}}(\mathbf{z})$ . We just need to substitute

$$\begin{aligned} w &= z^2 \\ z &= w^{1/2} \\ 2z dz &= dw \end{aligned} \tag{18}$$

in

$$f_{\mathbf{z}} dz = 2^{-(n/2)+1} [\Gamma(n/2)]^{-1} z^{n-1} e^{-z^2/2} dz \tag{19}$$

This gives

$$\begin{aligned} f_{\mathbf{z}} dz &= 2^{-n/2} [\Gamma(n/2)]^{-1} z^{n-2} e^{-z^2/2} 2z dz \\ &= 2^{-n/2} [\Gamma(n/2)]^{-1} w^{(n-2)/2} e^{-w/2} dw = f_{\mathbf{w}} dw \end{aligned} \tag{20}$$

The function

$$f_{\mathbf{w}}(w) = 2^{-n/2} [\Gamma(n/2)]^{-1} w^{(n-2)/2} e^{-w/2} \tag{21}$$

is the probability distribution function of the random variable  $\mathbf{w}$  which is called *chi-squared* ( $\chi^2$ ) and (21) is called the chi-squared distribution. We notice that the chi-squared distribution is defined for the random variable  $\mathbf{w}$  having values  $w \geq 0$  defined in equation (17).

## References

- [1] Helstrom, Carl W. (1984) *Probability and stochastic processes for engineers* Macmillan Publishing Company New York, Collier Macmillan Publishers London

Pawel Jan Piskorz (paweljs@gmail.com)

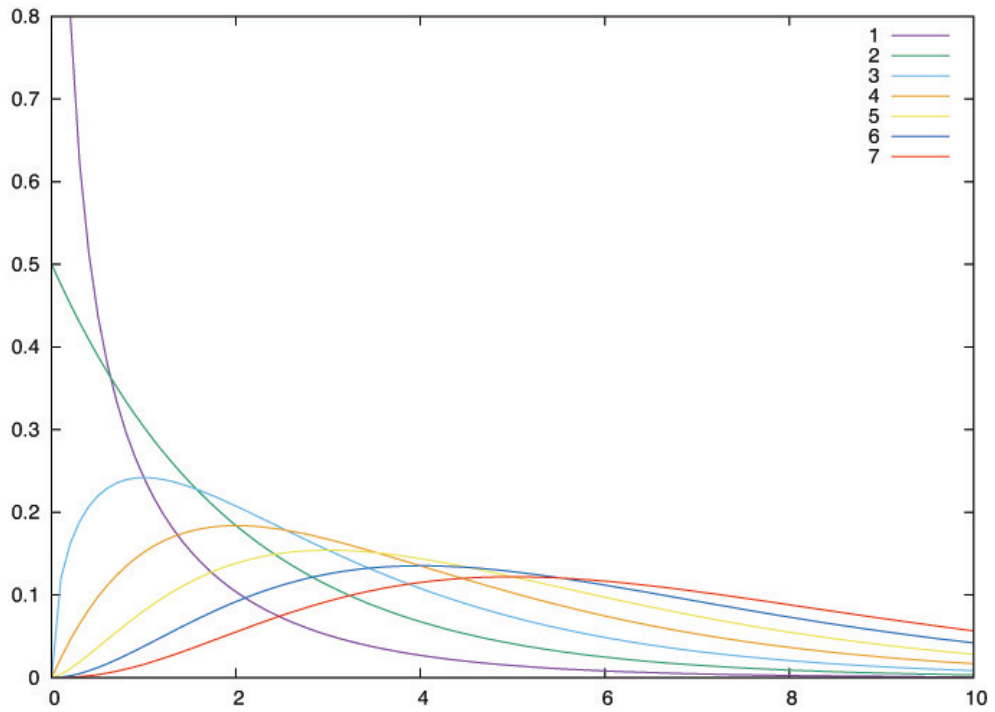


Figure 1: Plots of chi-squared probability density functions for  $n = 1, 2, \dots, 7$