Spherically symmetrical probability density functions and the chi-squared distribution

Let us have a spherically symmetrical probability density function

$$f_{\mathbf{x}}(x_1, x_2, \ldots, x_n) = p(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}) \tag{1}$$

everywhere in $\mathbb{R}^n$, where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ and $p(r)$ is a nonnegative function of the single variable r. To subject this joint density function to the normalization condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{x}}(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \cdots dx_n = 1 \tag{2}$$

we must evaluate its integral over all of $\mathbb{R}^n$. We can do so by utilizing the fact that the probability density function $f_{\mathbf{x}}(x_1, x_2, \ldots, x_n)$ is constant over a hyperspherical shell. The volume dV of a hyperspherical shell of radius r and thickness dr in n-dimensional space is proportional to $r^{n-1} dr$

$$dV = k_n r^{n-1}\, dr \tag{3}$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{x}}(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \cdots dx_n = k_n \int_0^{\infty} r^{n-1} p(r)\, dr \quad (4) \\ = 1 \end{aligned}$$

The constant k_n can be worked out by picking a particular joint density function for which both sides of equation (4) can be easily evaluated. A good function for that evaluation is a Gaussian function

$$f_{\mathbf{x}}(x_1, x_2, \ldots, x_n) = (2\pi)^{-n/2} e^{-(x_1^2 + x_2^2 + \cdots + x_n^2)/2} \tag{5}$$

for which

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1 + x_2 + \cdots + x_n)/2}\, dx_1 dx_2 \ldots dx_n \quad (6) \\ = \int_{-\infty}^{\infty} e^{-x_1/2}\, dx_1 \int_{-\infty}^{\infty} e^{-x_2/2}\, dx_2 \cdots \int_{-\infty}^{\infty} e^{-x_n/2}\, dx_n = (2\pi)^{n/2} \end{aligned}$$

and after changing the variables of integration

$$k_n \int_0^\infty r^{n-1} e^{-r^2/2}\, dr = (2\pi)^{n/2} \tag{7}$$

and if

$$f_{\mathbf{x}}(x_1, x_2, \ldots, x_n) = (2\pi)^{-n/2} e^{-(x_1+x_2+\cdots+x_n)/2} \tag{8}$$

then

$$k_n (2\pi)^{-n/2} \int_0^\infty r^{n-1} e^{-r^2/2}\, dr = 1 \tag{9}$$

and $p(r)$ corresponding to $f_{\mathbf{x}}(x_1, x_2, \ldots, x_n)$ is

$$p(r) = (2\pi)^{-n/2} e^{-r^2/2} \tag{10}$$

We now compute the integral in equation (9)

$$\int_0^\infty r^{n-1} e^{-r^2/2}\, dr = \left| \begin{array}{rcl} r^2/2 & = & u \\ rdr & = & du \\ r & = & (2u)^{1/2} \end{array} \right| = \int_0^\infty r^{n-2} e^{-r^2/2} r\, dr \tag{11}$$

$$= \int_0^\infty (2u)^{(n-2)/2} e^{-u}\, du = \frac{1}{2} 2^{n/2} \int_0^\infty u^{n/2-1} e^{-u}\, du = \frac{1}{2} 2^{n/2} \Gamma(n/2)$$

and we have

$$k_n (2\pi)^{-n/2} \frac{1}{2} 2^{n/2} \Gamma(n/2) = 1 \tag{12}$$

what gives

$$k_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{13}$$

and

$$f_{\mathbf{x}}\, dx = p(x) k_n x^{n-1}\, dx \tag{14}$$

$$f_{\mathbf{x}} = \frac{2\pi^{n/2} x^{n-1} p(x)}{\Gamma(n/2)} \tag{15}$$

what we can rewrite as

$$f_{\mathbf{z}} = 2^{-(n/2)+1} [\Gamma(n/2)]^{-1} z^{n-1} e^{-z^2/2} \tag{16}$$

for

$$\mathbf{z} = (\mathbf{x_1}^2 + \mathbf{x_2}^2 + \cdots + \mathbf{x_n}^2)^{1/2} \tag{17}$$

Now we introduce the variable

$$\mathbf{w} = \mathbf{z}^2 = \mathbf{x_1}^2 + \mathbf{x_2}^2 + \cdots + \mathbf{x_n}^2 \tag{18}$$

and the question is what is the probability distribution function $f_{\mathbf{w}}(\mathbf{w})$ if we know the function $f_{\mathbf{z}}(\mathbf{z})$. We just need to substitute

$$\begin{aligned} w &= z^2 \\ z &= w^{1/2} \\ 2z\,dz &= dw \end{aligned} \tag{19}$$

in

$$f_{\mathbf{z}}\,dz = 2^{-(n/2)+1}[\Gamma(n/2)]^{-1} z^{n-1} e^{-z^2/2}\,dz \tag{20}$$

This gives

$$\begin{aligned} f_{\mathbf{z}}\,dz &= 2^{-n/2}[\Gamma(n/2)]^{-1} z^{n-2} e^{-z^2/2} 2z\,dz \\ &= 2^{-n/2}[\Gamma(n/2)]^{-1} w^{(n-2)/2} e^{-w/2}\,dw = f_{\mathbf{w}}\,dw \end{aligned} \tag{21}$$

The function

$$f_{\mathbf{w}}(w) = 2^{-n/2}[\Gamma(n/2)]^{-1} w^{(n-2)/2} e^{-w/2} \tag{22}$$

is the probability distribution function of the random variable $\mathbf{w}$ which is called *chi-squared* (χ^2) and (22) is called the chi-squared distribution. We notice that the chi-squared distribution is defined for the random variable $\mathbf{w}$ having values $w \geq 0$ defined in equation (18).

References

[1] Helstrom, Carl W. (1984) *Probability and stochastic processes for engineers* Macmillan Publishing Company New York, Collier Macmillan Publishers London

Pawel Jan Piskorz (paweljs@gmail.com)

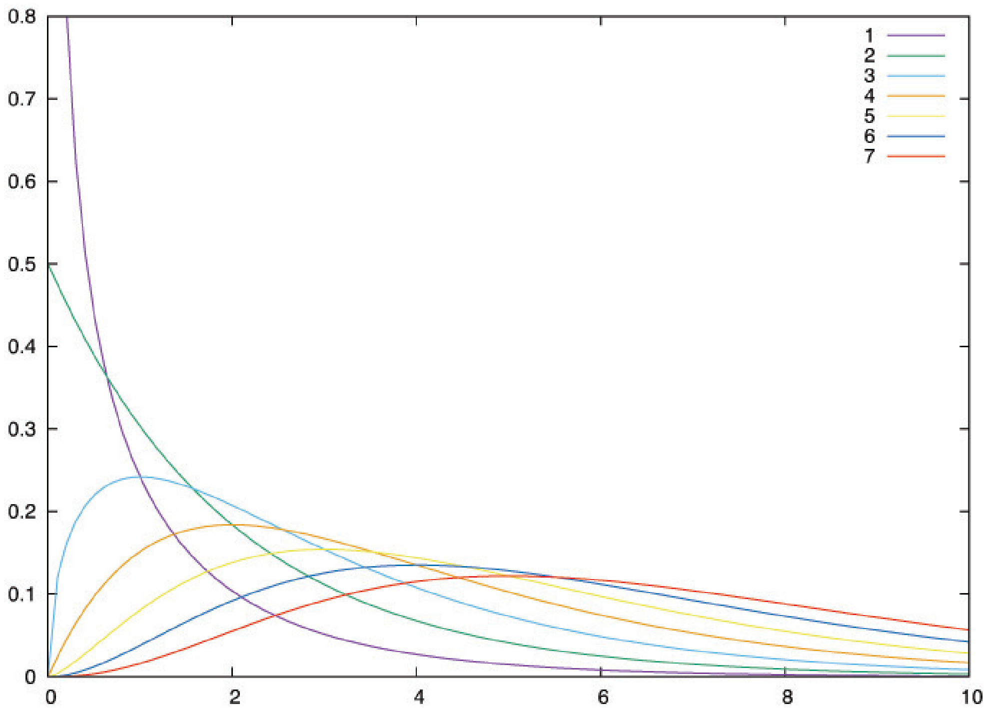

Figure 1: Plots of chi-squared probability density functions for $n = 1, 2, \ldots, 7$