Note on expected value of angular momentum in Quantum Mechanics

In [1] on page 91 we find the idea of using the formula for the sum of the squares of natural numbers to derive the average value of the angular momentum in quantum mechanics.

We have in quantum mechanics that the average value of the square of angular momentum $\langle \vec{m}^{\,2} \rangle$ has the following properties as the consequence of Pythagorean theorem and the equavalence of the three Cartesian coordinates in physical three dimensional space

$$\langle \vec{m}^{\,2} \rangle = \langle \vec{m}_x^2 + \vec{m}_y^2 + \vec{m}_z^2 \rangle = \langle \vec{m}_x^2 \rangle + \langle \vec{m}_y^2 \rangle + \langle \vec{m}_z^2 \rangle \tag{1}$$

$$\langle \vec{m}_x^2 \rangle = \langle \vec{m}_y^2 \rangle = \langle \vec{m}_z^2 \rangle \tag{2}$$

$$\langle \vec{m}^{\,2} \rangle = 3 \langle \vec{m}_z^2 \rangle \tag{3}$$

where $\langle \vec{m}_x^2 \rangle, \langle \vec{m}_y^2 \rangle, \langle \vec{m}_z^2 \rangle$ are the average squared components of the angular momentum along the axes x, y, and z, respectively.

We have also the following relation for the quantum number m (the magnetic quantum number) of the z-component of the angular momentum

$$\begin{aligned} m_z &= m\hbar \\ \vec{m}_z^2 &= m^2\hbar^2 \end{aligned} \tag{4}$$

where m assumes $2l + 1$ values $m = 0, \pm 1, \ldots, \pm l$. l is the angular momentum quantum number. In case of hydrogen atom l is the orbital angular momentum quantum number of the electron. Electron itself has an intrinsic angular momentum which is called the spin, and the spin angular momentum quantum number is denoted by s.

The mean value $\langle \vec{m}_z^2 \rangle$ is equal to

$$\langle \vec{m}_z^2 \rangle = \frac{1}{2l+1} \sum_{m=-l}^{l} m^2\hbar^2 \tag{5}$$

From the square pyramidal number formula which was computed in [2]

$$\sum_{m=0}^{l} m^2 = \frac{l(l+1)(2l+1)}{6} \tag{6}$$

and

$$\sum_{m=-l}^{l} m^2\hbar^2 = 2\sum_{m=0}^{l} m^2\hbar^2 = 2\frac{l(l+1)(2l+1)}{6}\hbar^2 = \frac{l(l+1)(2l+1)}{3}\hbar^2 \tag{7}$$

$$\langle \vec{m}_z^{\,2} \rangle = \frac{1}{2l+1}\frac{l(l+1)(2l+1)}{3}\hbar^2 = \frac{l(l+1)}{3}\hbar^2 \tag{8}$$

Therefrom

$$\langle \vec{m}^{\,2} \rangle = 3\frac{l(l+1)}{3}\hbar^2 \tag{9}$$

which gives the formula

$$\langle \vec{m}^{\,2} \rangle = l(l+1)\hbar^2 \tag{10}$$

The absolute value $|\vec{m}|$ of the angular momentum vector $\vec{m}$ is denoted by L, and we have the formula

$$L = \sqrt{l(l+1)}\hbar \tag{11}$$

The above formula is also valid for the fractional magnetic quantum numbers, like for the magnetic spin quantum number of the electron $s = 1/2$. There is also $2s+1$ orientations of the spin (intrinsic angular momentum) of the electron in the uniform external magnetic field, and the formula for the value of the intrinsic angular momentum S of the electron is

$$S = \sqrt{s(s+1)}\hbar \tag{12}$$

For $s = 1/2$ there are values of $m = \pm 1/2$. If we consider $s = 3/2$, we have $m = \pm 1/2, \pm 3/2$. m value of zero cannot occur because the maximum value of m must be equal to s and m values must differ by 1. Anonymous readers of the manuscript noticed that the derivation of equation (11) or (12) for the fractional values of m is also valid with use of the formula in (6).

References

[1] Alojzy Golebiewski, *Elementy mechaniki i chemii kwantowej*, Panstwowe Wydawnictwo Naukowe, Warszawa, 1982.

[2] Pawel J. Piskorz, *100.02 Square pyramidal numbers and the multiplication table*, The Mathematical Gazette, Volume 100, Issue 547, March 2016, pp. 108-111, doi:10.1017/mag.2016.11.

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