Angular momentum and Euler equations

The angular momentum $\boldsymbol{L}$ has in general its components L_x, L_y, L_z given by

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{1}$$

and they can be expressed as

$$L_x = I_{xx}\omega_x, L_y = I_{yy}\omega_y, L_z = I_{zz}\omega_z \tag{2}$$

if the coordinate axes are the main axes of inertia. We choose the coordinate system having axes x', y', z' along the main axes of inertia of the body. The versors are $\boldsymbol{i_{x'}}, \boldsymbol{i_{y'}}, \boldsymbol{i_{z'}}$. Angular velocity has now the components

$$\boldsymbol{\omega} = \omega_{x'}\boldsymbol{i_{x'}} + \omega_{y'}\boldsymbol{i_{y'}} + \omega_{z'}\boldsymbol{i_{z'}} \tag{3}$$

and the angular momentum

$$\boldsymbol{L} = I_{x'}\omega_{x'}\boldsymbol{i_{x'}} + I_{y'}\omega_{y'}\boldsymbol{i_{y'}} + I_{z'}\omega_{z'}\boldsymbol{i_{z'}} \tag{4}$$

The products

$$\begin{aligned} I_{x'}\omega_{x'}\boldsymbol{i_{x'}} &= L_{x'}\boldsymbol{i_{x'}} \\ I_{y'}\omega_{y'}\boldsymbol{i_{y'}} &= L_{y'}\boldsymbol{i_{y'}} \\ I_{z'}\omega_{z'}\boldsymbol{i_{z'}} &= L_{z'}\boldsymbol{i_{z'}} \end{aligned} \tag{5}$$

are the components of the angular momentum vector in the coordinate system which rotates together with the body.

Because in general

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{6}$$

we also have

$$\frac{d\boldsymbol{i_{x'}}}{dt} = \boldsymbol{\omega} \times \boldsymbol{i_{x'}},\ \frac{d\boldsymbol{i_{y'}}}{dt} = \boldsymbol{\omega} \times \boldsymbol{i_{y'}},\ \frac{d\boldsymbol{i_{z'}}}{dt} = \boldsymbol{\omega} \times \boldsymbol{i_{z'}} \tag{7}$$

The derivative of angular momentum $\boldsymbol{L}$

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= I_{x'}\frac{d\omega_{x'}}{dt}\boldsymbol{i_{x'}} + I_{y'}\frac{d\omega_{y'}}{dt}\boldsymbol{i_{y'}} + I_{z'}\frac{d\omega_{z'}}{dt}\boldsymbol{i_{z'}} \\ &\quad + \boldsymbol{\omega} \times (I_{x'}\omega_{x'}\boldsymbol{i_{x'}} + I_{y'}\omega_{y'}\boldsymbol{i_{y'}} + I_{z'}\omega_{z'}\boldsymbol{i_{z'}}) \\ &= I_{x'}\frac{d\omega_{x'}}{dt}\boldsymbol{i_{x'}} + I_{y'}\frac{d\omega_{y'}}{dt}\boldsymbol{i_{y'}} + I_{z'}\frac{d\omega_{z'}}{dt}\boldsymbol{i_{z'}} + \boldsymbol{\omega} \times \boldsymbol{L'} \end{aligned} \tag{8}$$

We have then

$$\boldsymbol{M} = \frac{d\boldsymbol{L}}{dt} = \left(\frac{d\boldsymbol{L}}{dt}\right)' + \boldsymbol{\omega} \times \boldsymbol{L'} \tag{9}$$

and the terms which contribute to $(\frac{dL}{dt})'$ are the components of the angular momentum derivative in the moving coordinate system having versors $\boldsymbol{i_{x'}}, \boldsymbol{i_{y'}}, \boldsymbol{i_{z'}}$. We can calculate

$$\boldsymbol{\omega} \times \boldsymbol{L'} = \begin{vmatrix} \boldsymbol{i}_{x'} & \boldsymbol{i}_{y'} & \boldsymbol{i}_{z'} \\ \omega_{x'} & \omega_{y'} & \omega_{z'} \\ I_{x'}\omega_{x'} & I_{y'}\omega_{y'} & I_{z'}\omega_{z'} \end{vmatrix} \tag{10}$$

Taking into account equations (8), (9) and (10) we receive the system of three scalar equations

$$\begin{aligned} I_{x'}\frac{d\omega_{x'}}{dt} + (I_{z'} - I_{y'})\omega_{y'}\omega_{z'} &= M_{x'} \\ I_{y'}\frac{d\omega_{y'}}{dt} + (I_{x'} - I_{z'})\omega_{x'}\omega_{z'} &= M_{y'} \\ I_{z'}\frac{d\omega_{z'}}{dt} + (I_{y'} - I_{x'})\omega_{x'}\omega_{y'} &= M_{z'} \end{aligned} \tag{11}$$

The equations of the system (11) are called *Euler equations*.

Pawel Jan Piskorz (paweljs@gmail.com)