Note on triangle inequality

x, y are vectors in n-dimensional space of the set of real numbers $\mathbb{R}$ what we write as $x, y \in \mathbb{R}^n$. In other words

$$x = \sum_{i=1}^{n} x_i \mathbf{e}_i \tag{1}$$

$$y = \sum_{j=1}^{n} y_j \mathbf{e}_j$$

where $\mathbf{e}_i$ is the i-th basis vector, and x_i is the i-th component of the vector x. If basis vectors are orthonormal then we have

$$x \cdot y = \left(\sum_{i=1}^{n} x_i \mathbf{e}_i \right) \cdot \left(\sum_{j=1}^{n} y_j \mathbf{e}_j \right) = \sum_{i=1}^{n} x_i y_i \tag{2}$$

because the products

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{3}$$

We have for all vectors

$$(x + y)^2 = x^2 + y^2 + 2x \cdot y \tag{4}$$

Then

$$(x + y)^2 = |(x + y)^2| = |x^2| + |y^2| + 2|x \cdot y| \tag{5}$$

$$= |x|^2 + |y|^2 + 2|x \cdot y| \leq |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2 \tag{6}$$

because of the Cauchy-Schwarz inequality

$$|x \cdot y| \leq |x||y| \tag{7}$$

In inequality (7) the equality holds for linearly dependent vectors x and y.

As an intermediate result we have then

$$(x + y)^2 \leq (|x| + |y|)^2 \tag{8}$$

Taking square root of both sides of the inequality (8)

$$\sqrt{(x + y)^2} \leq \sqrt{(|x| + |y|)^2} \tag{9}$$

we receive

$$|x + y| \leq ||x| + |y|| \tag{10}$$

what is equivalent to

$$|x + y| \leq |x| + |y| \tag{11}$$

The inequality (11) is called a triangle inequality. $|x|$ and $|y|$ are the lengths of the two vectors x and y, and $|x + y|$ is the length of the sum of vectors x and y. It is a property of sum of any vectors x and y such that $x, y \in \mathbb{R}^n$, i.e. vectors x and y in n dimensions. It is a weak inequality, i.e. inequality having the sign $\leq$ and it also takes into account the general situation when the vectors x and y are linearly dependent what was assumed when the Cauchy-Schwarz inequality (7) was used to derive the inequalty (11). The situation when the vectors x and y are linearly independent corresponds to the strong inequality, i.e. inequality having the sign $<$

$$|x + y| < |x| + |y| \tag{12}$$

The name triangle inequality of the inequality (11) comes from talking about the linearly independent vectors x, y and $x+y$ in two dimensions, when these vectors lie along the sides of triangle in such way that x and y determine two sides of the triangle and the sum of vectors $x+y$ determines the third side of triangle (see Figure 1). Then the values of $|x|$ and $|y|$ are the lengths of the two sides of a triangle along the vectors x and y, and the value of $|x + y|$ is the length of the third side of the triangle along the vector $x + y$ The strong inequality (12) is valid in general in n dimensions for linearly independent vectors $x, y \in \mathbb{R}^n$.

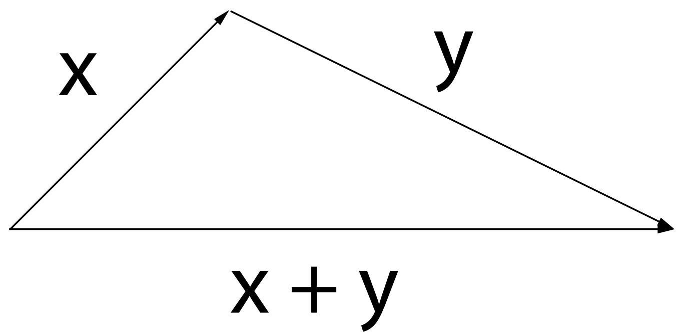

Figure 1: Vectors x and y determine two sides, and the vector $x+y$ determines the third side of a triangle

From inequality (12) we see that in particular for any triangle in two dimensions always the sum of the lengths of the two sides is greater than the length of the third side what we can express for any two-dimensional triangle having the lengths of the three sides equal to a, b, c, respectively as

$$\begin{aligned} a &< b + c \\ b &< c + a \\ c &< a + b \end{aligned} \tag{13}$$

The inequalities (13) are simple statements about a simple gemetrical figure which is a triangle. In this note the use of Cauchy-Schwarz inequality was required to prove these inequalities and the general triangle inequality (11).

References

[1] Spivak, M. (1998) *Calculus on Manifolds* Westview Press

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