Arctan(x) and the improper integral of Sin(x)/x

Let I denotes the integral

$$I = \int_0^\infty \frac{\sin(\mu x)}{x} e^{-ax}\, dx \tag{1}$$

Then

$$\frac{\partial I}{\partial \mu} = \int_0^\infty \frac{x\cos(\mu x)}{x} e^{-ax}\, dx = \int_0^\infty \cos(\mu x) e^{-ax}\, dx \tag{2}$$

Because (see Appendix A)

$$\int_0^\infty \cos(\mu x) e^{-ax}\, dx = \frac{a}{\mu^2 + a^2} \tag{3}$$

we have

$$\frac{\partial I}{\partial \mu} = \frac{a}{\mu^2 + a^2} \tag{4}$$

We notice that

$$I(\mu = 0) = 0 \tag{5}$$

and

$$I(\mu = 1) = \int_0^\infty \frac{\sin(x)}{x} e^{-ax}\, dx \tag{6}$$

Because of equations (5) and (6) we can integrate equation (4) calculating

$$\begin{aligned} I(\mu = 1) - I(\mu = 0) = \int_0^\infty \frac{\sin(x)}{x} e^{-ax}\, dx &= \int_0^1 \frac{a}{\mu^2 + a^2}\, d\mu \\ &= \arctan\left(\frac{1}{a}\right) \end{aligned} \tag{7}$$

Then we can write

$$I(\mu = 1, a) = \int_0^\infty \frac{sin(x)}{x} e^{-ax}\, dx = \arctan\left(\frac{1}{a}\right) \tag{8}$$

and we have

$$\lim_{a \to 0} I(\mu = 1, a) = \int_0^\infty \frac{sin(x)}{x}\,dx = \arctan(\infty) = \frac{\pi}{2} \tag{9}$$

and

$$\int_{-\infty}^\infty \frac{sin(x)}{x}\,dx = \pi \tag{10}$$

Appendix A Solving integral $\int_0^\infty \cos(\mu x) e^{-ax}\,dx$

$$\int e^{-ax}\cos(\mu x)\,dx = \left| \begin{array}{rcl} \mu x & = & v \\ dx & = & \frac{1}{\mu}dv \\ x & = & \frac{v}{\mu} \end{array} \right| = \frac{1}{\mu}\int e^{-\frac{a}{\mu}v}\cos(v)\,dv \tag{11}$$

$$= \frac{1}{\mu}\frac{e^{-\frac{a}{\mu}v}}{1+\frac{a^2}{\mu^2}}\Big(\sin(v) - \frac{a}{\mu}\cos(v)\Big) = L(v)$$

$$L(v \to \infty) = 0 \tag{12}$$

$$L(v = 0) = -\frac{a}{\mu^2 + a^2} \tag{13}$$

$$\int_0^\infty e^{-ax}\cos(\mu x)\,dx = L(v \to \infty) - L(v = 0) = \frac{a}{\mu^2 + a^2} \tag{14}$$

Appendix B Solving integral $\int_0^1 \frac{a}{\mu^2+a^2}\,d\mu$

$$\int \frac{a}{\mu^2 + a^2}\,d\mu = a\int \frac{1}{\mu^2 + a^2}\,d\mu = \frac{a}{a^2}\int \frac{1}{\left(\frac{\mu}{a}\right)^2 + 1}\,d\mu \tag{15}$$

$$= \frac{1}{a}\int \frac{1}{\left(\frac{\mu}{a}\right)^2 + 1}\,d\mu = \left| \begin{array}{rcl} \frac{\mu}{a} & = & z \\ d\mu & = & adz \end{array} \right|$$

$$= \int \frac{1}{z^2+1}\,dz = \arctan(z) + C = \arctan\left(\frac{\mu}{a}\right) + C$$

Then

$$\int_0^1 \frac{a}{\mu^2 + a^2}\,d\mu = \arctan\left(\frac{1}{a}\right) - \arctan\left(\frac{0}{a}\right) = \arctan\left(\frac{1}{a}\right) \tag{16}$$

Appendix C Solving integrals $\int e^{ax} \sin x \, dx$ and $\int e^{ax} \cos x \, dx$

Let

$$I_1 = \int e^{ax} \sin x \, dx \tag{17}$$

and

$$I_2 = \int e^{ax} \cos x \, dx \tag{18}$$

Now we integrate by parts applying the formula:

$$\int u'v dx = uv - \int uv' dx \tag{19}$$

For the integral I_1 we have

$$\begin{aligned} \int e^{ax} \sin x \, dx = \left| \begin{array}{ccc} e^{ax} & = & u \\ \sin x & = & v' \end{array} \right| \left| \begin{array}{ccc} u' & = & ae^{ax} \\ v & = & -\cos x \end{array} \right| = \\ = -e^{ax} \cos x + a \int e^{ax} \cos x \, dx \end{aligned} \tag{20}$$

and for the integral I_2

$$\begin{aligned} \int e^{ax} \cos x \, dx = \left| \begin{array}{ccc} e^{ax} & = & u \\ \cos x & = & v' \end{array} \right| \left| \begin{array}{ccc} u' & = & ae^{ax} \\ v & = & \sin x \end{array} \right| = \\ = e^{ax} \sin x - a \int e^{ax} \sin x \, dx \end{aligned} \tag{21}$$

We arrive at the following system of equations:

$$\begin{array}{rcrcl} I_1 & - & aI_2 & = & -e^{ax} \cos x \\ aI_1 & + & I_2 & = & e^{ax} \sin x \end{array} \tag{22}$$

The determinants D, D_1 and D_2 are

$$D = \left| \begin{array}{cr} 1 & -a \\ a & 1 \end{array} \right| = 1 + a^2 \tag{23}$$

$$\begin{aligned} D_1 = \left| \begin{array}{cr} -e^{ax} \cos x & -a \\ e^{ax} \sin x & 1 \end{array} \right| \\ = -e^{ax} \cos x + ae^{ax} \sin x = e^{ax}(-\cos x + a \sin x) \end{aligned} \tag{24}$$

and

$$D_2 = \begin{vmatrix} 1 & -e^{ax}\cos x \\ a & e^{ax}\sin x \end{vmatrix} \tag{25}$$

$$= e^{ax}\sin x + ae^{ax}\cos x = e^{ax}(\sin x + a\cos x)$$

The integrals I_1 and I_2 become

$$\begin{aligned} \int e^{ax}\sin x\,dx &= \frac{e^{ax}}{1+a^2}(-\cos x + a\sin x) + C \\ \int e^{ax}\cos x\,dx &= \frac{e^{ax}}{1+a^2}(\sin x + a\cos x) + C \end{aligned}$$

Appendix D arctan(x) derivative

We have

$$y = \tan x \tag{26}$$

and

$$x = \arctan y \tag{27}$$

$$\frac{dy}{dx} = \frac{d\tan x}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{\tan^2 x + 1}{1} \tag{28}$$

Therefrom

$$\frac{dy}{dx} = \frac{y^2+1}{1} \tag{29}$$

and

$$\frac{dx}{dy} = \frac{d\arctan y}{dy} = \frac{1}{y^2+1} \tag{30}$$

what can be rewritten as

$$\frac{d\arctan x}{dx} = \frac{1}{x^2+1} \tag{31}$$

We have then

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C \tag{32}$$

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