Sum of Cosines and the Improper Integral of Sin(x)/x

We will introduce the derivation of the formula for the series of cosines and sines following the proof presented in [1]. Then we will work on the derived sum of cosines to compute the improper integral of $\frac{\sin x}{x}$ which will be our main result. We will also compute the improper integral of $\frac{\sin^2 x}{x^2}$.

Sum of Geometric Series and the Sum of $e^{ik\theta}$ We have the n-th partial sum s_n of the geometric series with the k-th term ar^{k-1} as equal to

$$s_n = \sum_{k=1}^{n} ar^{k-1} = a\frac{1-r^n}{1-r} \tag{1}$$

for $r \neq 1$. This result can be derived by computing the difference between s_n and rs_n which equals to

$$s_n - rs_n = a - ar^n \tag{2}$$

Using the Euler formula for $e^{ik\theta}$ we can write

$$\sum_{k=1}^{n}(\cos k\theta + i\sin k\theta) = \sum_{k=1}^{n} e^{ik\theta} = e^{i\theta}\sum_{k=1}^{n} e^{i(k-1)\theta} \tag{3}$$

Now we notice that the last sum in the equation (3) is the sum s_n from the equation (1) with $a = 1$ and $r = e^{i\theta}$, so we can write

$$\sum_{k=1}^{n}(\cos k\theta + i\sin k\theta) = e^{i\theta}\frac{e^{in\theta}-1}{e^{i\theta}-1} \tag{4}$$

for $e^{i\theta} \neq 1$. Further, following [1], we will derive a useful identity

$$e^{i\alpha} - 1 = e^{i\alpha/2}(e^{i\alpha/2} - e^{-i\alpha/2}) = e^{i\alpha/2}2i\sin(\alpha/2) \tag{5}$$

Applying this identity to the result from the equation (4) we receive

$$e^{i\theta}\frac{e^{in\theta}-1}{e^{i\theta}-1} = e^{i\theta}\frac{e^{in\theta/2}2i\sin(n\theta/2)}{e^{i\theta/2}2i\sin(\theta/2)} = e^{i(n+1)\theta/2}\frac{\sin(n\theta/2)}{\sin(\theta/2)} \tag{6}$$

Sum of Cosine and Sine Series Now we can separate the last expression from equation (6) into the real and imaginary parts, and we obtain the expressions for the sum of cosines and the sum of sines presented in the equations (7) and (8), respectively

$$\sum_{k=1}^{n} \cos k\theta = \cos\left[(n+1)\theta/2\right] \frac{\sin(n\theta/2)}{\sin(\theta/2)} \tag{7}$$

$$\sum_{k=1}^{n} \sin k\theta = \sin\left[(n+1)\theta/2\right] \frac{\sin(n\theta/2)}{\sin(\theta/2)} \tag{8}$$

In the sum of cosines let's have a look at the

$$\cos\left[(n+1)\theta/2\right] \sin(n\theta/2) \tag{9}$$

Rewriting the above as $\sin\alpha\cos\beta$ and using the formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{10}$$

and

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{11}$$

we arrive at

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right] \tag{12}$$

With

$$\alpha = n\theta/2 \tag{13}$$

$$\beta = (n+1)\theta/2 \tag{14}$$

$$\alpha+\beta = n\theta/2 + (n+1)\theta/2 = (2n+1)\theta/2 \tag{15}$$

$$\alpha-\beta = n\theta/2 - (n+1)\theta/2 = -\theta/2 \tag{16}$$

we get

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin\left[(2n+1)\theta/2\right] - \sin(\theta/2)\right) \tag{17}$$

what leads to

$$\sum_{k=1}^{n} \cos k\theta = \frac{1}{2}\,\frac{\sin\left[(2n+1)\theta/2\right] - \sin(\theta/2)}{\sin(\theta/2)} \tag{18}$$

and to

$$\frac{1}{2} + \sum_{k=1}^{n} \cos k\theta = \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)} \tag{19}$$

Improper Integral of $\frac{\sin x}{x}$ Integrating both sides of the equation (19) over θ in $[-\pi, \pi]$ we have

$$\int_{-\pi}^{\pi} \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}\, d\theta = \pi \tag{20}$$

because the cosines integrals equal zero.

Let's substitute $\theta = x/n$, $d\theta = dx/n$ for $n = 1, 2, \ldots$

$$\int_{-\pi}^{\pi} \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}\, d\theta = \int_{-n\pi}^{n\pi} \frac{\sin\left[(2n+1)x/2n\right]}{2n\sin(x/2n)}\, dx \tag{21}$$

In particular, for $n \to \infty$

$$\lim_{n\to\infty} \int_{-n\pi}^{n\pi} \frac{\sin\left[(2n+1)x/2n\right]}{2n\sin(x/2n)}\, dx = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi \tag{22}$$

We have computed separately the limits of the numerator

$$\lim_{n\to\infty} \sin\left[(2n+1)x/2n\right] = \sin x \tag{23}$$

because

$$\lim_{n\to\infty} \frac{2n+1}{2n} = 1 \tag{24}$$

and the denominator

$$\lim_{n\to\infty} 2n\sin(x/2n) = \lim_{n\to\infty} x\frac{\sin(x/2n)}{x/2n} = x \lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = x \tag{25}$$

Improper Integral of $\frac{\sin^2 x}{x^2}$ Having the result from the equation (22) we can also attempt to compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx \tag{26}$$

Integrating by parts we obtain

$$\int_{-\infty}^{\infty} x^{-2}\sin^2 x\, dx = \begin{vmatrix} x^{-2} & = & u' \\ \sin^2 x & = & v \end{vmatrix}\begin{vmatrix} u & = & -x^{-1} \\ v' & = & 2\sin x\cos x = \sin 2x \end{vmatrix} = \tag{27}$$

$$\left[-x^{-1}\sin^2 x\right]_{x=-\infty}^{x=\infty} + \int_{-\infty}^{\infty} \frac{\sin 2x}{x}\, dx$$

We notice that

$$\left[-x^{-1}\sin^2 x\right]_{x=-\infty}^{x=\infty} = 0 \tag{28}$$

and substituting in the integral

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x}\, dx \tag{29}$$

$u = 2x$, $du = 2dx$ we receive

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \int_{-\infty}^{\infty} \frac{\sin u}{u}\, du \tag{30}$$

and our second result is

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi \tag{31}$$

References

[1] Hirst, K.E. (1995) *Numbers, Sequences and Series* London, Sydney, Auckland: Arnold, ISBN: 0 340 61043 3