Sum of Cosines Integration

We have

$$\sum_{k=1}^{n} \cos k\theta = \cos\left[(n+1)\theta/2\right] \frac{\sin(n\theta/2)}{\sin(\theta/2)}$$

and

$$\sum_{k=1}^{n} \sin k\theta = \sin\left[(n+1)\theta/2\right] \frac{\sin(n\theta/2)}{\sin(\theta/2)}$$

Sum of Cosines Integration

In the sum of cosines let's have a look at the

$$\cos\left[(n+1)\theta/2\right]\sin(n\theta/2)$$

Rewriting the above as $\sin\alpha\cos\beta$ and using the formulas

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

and

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

Sum of Cosines Integration

we arrive at

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

With

$$\alpha = n\theta/2$$

and

$$\beta = (n+1)\theta/2$$

$$\alpha+\beta = n\theta/2 + (n+1)\theta/2 = (2n+1)\theta/2$$

$$\alpha-\beta = n\theta/2 - (n+1)\theta/2 = -\theta/2$$

Sum of Cosines Integration

we get

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin\left[(2n+1)\theta/2\right] - \sin(\theta/2)\right)$$

what leads to

$$\sum_{k=1}^{n}\cos k\theta = \frac{1}{2}\,\frac{\sin\left[(2n+1)\theta/2\right] - \sin(\theta/2)}{\sin(\theta/2)}$$

Sum of Cosines Integration

The sum of cosine series becomes

$$\frac{1}{2} + \sum_{k=1}^{n} \cos k\theta = \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}$$

Integrating both sides over θ in $[-\pi, \pi]$ we have

$$\int_{-\pi}^{\pi} \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}\, d\theta = \pi$$

Sum of Cosines Integration

Let's substitute $\theta = \alpha/n$, $d\theta = d\alpha/n$

$$\int_{-\pi}^{\pi} \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}\, d\theta = \int_{-n\pi}^{n\pi} \frac{\sin\left[(2n+1)\alpha/2n\right]}{2n\sin(\alpha/2n)}\, d\alpha$$

In particular, for $n \to \infty$

$$\lim_{n\to\infty} \int_{-n\pi}^{n\pi} \frac{\sin\left[(2n+1)\alpha/2n\right]}{2n\sin(\alpha/2n)}\, d\alpha = \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\, d\alpha = \pi$$