The Poisson Distribution

If value x of random variable X is binomially distributed then

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \tag{1}$$

and $E(X) = np$. Let $\lambda = np$ so that $p = \lambda/n$. p is the probability of success in binomial distribution denoting the ratio

$$p = \tag{2}$$

$$\frac{\lambda}{n} = \frac{\text{number of successes, i.e. investigated events e.g. per unit time}}{\text{number of all events, i.e. successes and failures e.g. per unit time}}$$

Then (1) becomes

$$\begin{aligned} P(X = x) &= \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \qquad (3) \\ &= \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \\ &= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!\, n^x} \lambda^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \\ &= \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{x-1}{n}\right)}{x!} \lambda^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \end{aligned}$$

Now as $n \to \infty$,

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{x-1}{n}\right) \to 1 \tag{4}$$

while

$$\left(1 - \frac{\lambda}{n}\right)^{n-x} = \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \to (e^{-\lambda})(1) \tag{5}$$

using the result from calculus that

$$\lim_{n\to\infty} \left(1 + \frac{u}{n}\right)^n = e^u \tag{6}$$

It follows that when $n \to \infty$ but λ stays fixed (i.e. $p \to 0$)

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \tag{7}$$

which is the Poisson distribution of value x of random variable X.