Note on mean and variance of sum of random variables

Consider a problem of finding the mean and variance of a sum of independent random variables. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be set of independent random variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ and let

$$\mathbf{z} = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n \tag{1}$$

We have then

$$\mu_{\mathbf{z}} = E(\mathbf{z}) = \sum_{i=1}^{n} E(\mathbf{x}_i) = \sum_{i=1}^{n} \mu_i \tag{2}$$

Next we have

$$\mathbf{z} - \mu_{\mathbf{z}} = (\mathbf{x}_1 - \mu_1) + (\mathbf{x}_2 - \mu_2) + \cdots + (\mathbf{x}_n - \mu_n) \tag{3}$$

Then

$$(\mathbf{z} - \mu_{\mathbf{z}})^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathbf{x}_i - \mu_i)(\mathbf{x}_j - \mu_j) \tag{4}$$

and

$$E(\mathbf{z} - \mu_{\mathbf{z}})^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} E(\mathbf{x}_i - \mu_i)(\mathbf{x}_j - \mu_j) \tag{5}$$

We have

$$E(\mathbf{x}_i - \mu_i)(\mathbf{x}_j - \mu_j) = E(\mathbf{x}_i - \mu_i)E(\mathbf{x}_j - \mu_j), \;\; i \neq j \tag{6}$$

but $E(\mathbf{x}_i - \mu_i) = 0$ so the equation (5) reduces to

$$E(\mathbf{z} - \mu_{\mathbf{z}})^2 = \sum_{i=1}^{n} E(\mathbf{x}_i - \mu_i)^2 \tag{7}$$

Since $E(\mathbf{x}_i - \mu_i)^2 = \sigma_i^2$, this result can be expressed as

$$\sigma_{\mathbf{z}}^2 = \sum_{i=1}^{n} \sigma_i^2 \tag{8}$$

References

[1] Paul G. Hoel, *Introduction to Mathematical Statistics* John Wiley & Sons, Inc., New York, London, Sidney, 1966, page 136.

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