Locus example: lemniscate of Bernoulli

Locus of point $P(x, y)$ satisfying given condition

$$F(x, y) = f(x, y) \tag{1}$$

we call the set of all points coordinates of which are fulfilling the equation (1). Equation (1) we call the *equation of locus.*

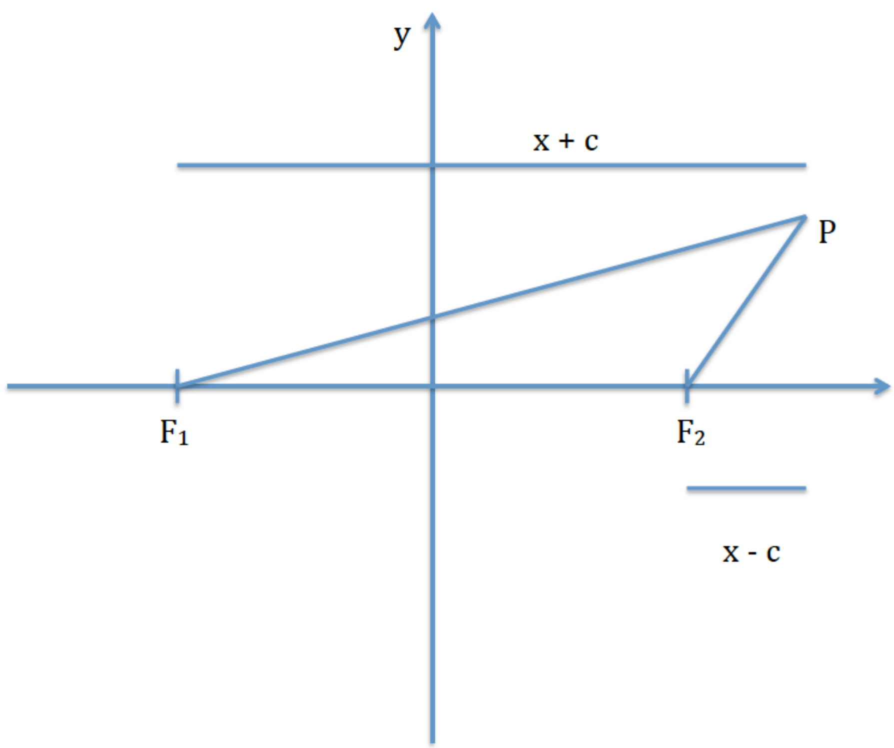

Figure 1: We have the following point coordinates: $P(x, y)$, $F_1(-c, 0)$, and $F_2(c, 0)$. Therefore from Pythagorean theorem we have the distance $|\overline{PF_1}|^2 = (x + c)^2 + y^2$ and $|\overline{PF_2}|^2 = (x - c)^2 + y^2$.

The system of coordinates we choose in the way that the axis Ox contains the vector $\overrightarrow{F_1F_2}$ and it has the same direction with this vector. The origin of the system of coordinates is in the middle of the segment $\overline{F_1F_2}$.

Let us find locus of the point $P(x, y)$ for which the product of the distance from two given points F_1 and F_2 separated by the distance $2c$ is constant and equal to c^2.

$$|\overline{PF_1}||\overline{PF_2}| = c^2 \tag{2}$$

Let us take power of two of the above equation (2)

$$|\overline{PF_1}|^2|\overline{PF_2}|^2 = c^4 \tag{3}$$

what is equivalent to

$$[(x+c)^2 + y^2][(x-c)^2 + y^2] = c^4 \tag{4}$$

what after rearrangements gives

$$\begin{aligned}[(x^2 + y^2 + c^2) + 2cx][(x^2 + y^2 + c^2) - 2cx] &= c^4 \\ (x^2 + y^2 + c^2)^2 - 4c^2x^2 &= c^4 \\ x^4 + 2x^2y^2 + y^4 - 2c^2x^2 + 2c^2y^2 & \end{aligned} \tag{5}$$

and we finally obtain

$$(x^2 + y^2)^2 = 2c^2(x^2 - y^2) \tag{6}$$

which is the equation of locus we were trying to find. The equation (6) is being called the equation of Bernoulli lemniscate.

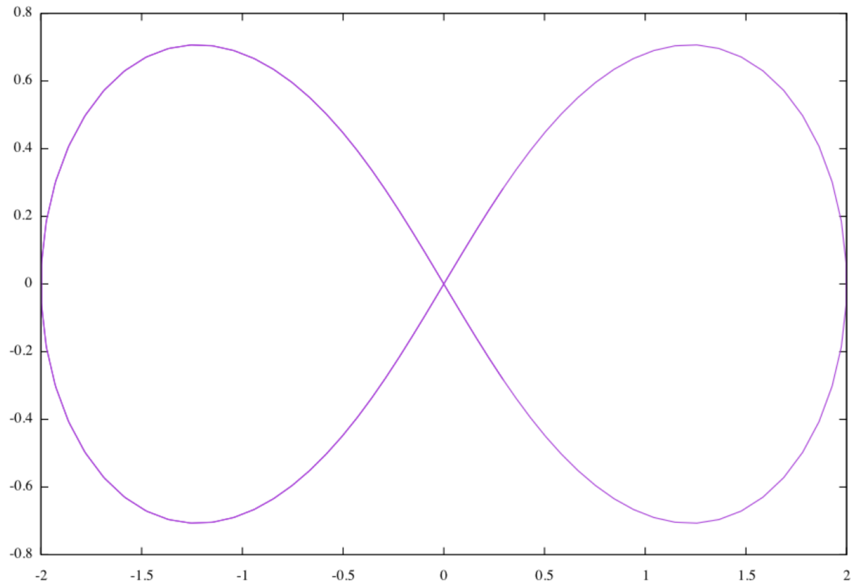

Figure 2: Lemniscate of Bernoulli example.

References

[1] Swietoslaw Romanowski and Wlodzimierz Wrona (1967) *Matematyka wyzsza dla studiow technicznych* Warszawa, Panstwowe Wydawnictwo Naukowe

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