Solving the integral $\int \frac{\ln(\cos x)}{\cos^2 x}\,dx$

$$\int \frac{\ln(\cos x)}{\cos^2 x}\,dx = \left|\begin{array}{ccc} \ln(\cos x) & = & f \\ 1/(\cos^2 x) & = & g' \end{array}\right|\left|\begin{array}{ccc} f' & = & -\tan x \\ g & = & \tan x \end{array}\right| \tag{1}$$
$$= (\tan x)\ln(\cos x) + \int \tan^2(x)\,dx$$

$$\int \tan^2(x)\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx - \int dx \tag{2}$$

$$\int \tan^2 x\,dx = \tan x - x + C \tag{3}$$

We have then

$$\int \frac{\ln(\cos x)}{\cos^2 x}\,dx = (\tan x)\ln(\cos x) + \tan x - x + C \tag{4}$$

We may verify the result by calculating the derivative of $(\tan x)\ln(\cos x) + \tan x - x + C$

$$((\tan x)\ln(\cos x) + \tan x - x + C)' = -x' + ((\tan x)(1+\ln(\cos x)))' \tag{5}$$
$$= \frac{1}{\cos^2 x}(1+\ln(\cos x)) + \tan x(\frac{1}{\cos x}(-\sin x)) - 1$$
$$= \frac{1}{\cos^2 x}(1+\ln(\cos x)) + \tan x(-\tan x) - 1$$
$$= \frac{1}{\cos^2 x}(1+\ln(\cos x)) - \tan^2 x - 1$$
$$= \frac{1}{\cos^2 x} + \frac{\ln(\cos x)}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}$$
$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} + \frac{\ln(\cos x)}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}$$
$$= \frac{\ln(\cos x)}{\cos^2 x}$$

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