Solving the Integrals $\int e^{ax} \sin x \, dx$ and $\int e^{ax} \cos x \, dx$

Let

$$I_1 = \int e^{ax} \sin x \, dx \tag{1}$$

and

$$I_2 = \int e^{ax} \cos x \, dx \tag{2}$$

Now we integrate by parts applying the formula:

$$\int u'v dx = uv - \int uv' dx \tag{3}$$

For the integral I_1 we have

$$\begin{aligned} \int e^{ax} \sin x \, dx &= \left| \begin{array}{ccc} e^{ax} & = & u \\ \sin x & = & v' \end{array} \right| \left| \begin{array}{ccc} u' & = & ae^{ax} \\ v & = & -\cos x \end{array} \right| = \\ &= -e^{ax} \cos x + a \int e^{ax} \cos x \, dx \end{aligned} \tag{4}$$

and for the integral I_2

$$\begin{aligned} \int e^{ax} \cos x \, dx &= \left| \begin{array}{ccc} e^{ax} & = & u \\ \cos x & = & v' \end{array} \right| \left| \begin{array}{ccc} u' & = & ae^{ax} \\ v & = & \sin x \end{array} \right| = \\ &= e^{ax} \sin x - a \int e^{ax} \sin x \, dx \end{aligned} \tag{5}$$

We arrive at the following system of equations:

$$\begin{array}{ccccc} I_1 & - & aI_2 & = & -e^{ax} \cos x \\ aI_1 & + & I_2 & = & e^{ax} \sin x \end{array} \tag{6}$$

The determinants D, D_1 and D_2 are

$$D = \left| \begin{array}{cc} 1 & -a \\ a & 1 \end{array} \right| = 1 + a^2 \tag{7}$$

$$D_1 = \left| \begin{array}{cc} -e^{ax} \cos x & -a \\ e^{ax} \sin x & 1 \end{array} \right| = -e^{ax} \cos x + ae^{ax} \sin x = e^{ax}(-\cos x + a \sin x) \tag{8}$$

and

$$D_2 = \left| \begin{array}{cc} 1 & -e^{ax} \cos x \\ a & e^{ax} \sin x \end{array} \right| = e^{ax} \sin x + ae^{ax} \cos x = e^{ax}(\sin x + a \cos x) \tag{9}$$

The integrals I_1 and I_2 become

$$\begin{aligned} \int e^{ax} \sin x \, dx &= \frac{e^{ax}}{1 + a^2}(-\cos x + a \sin x) + C \\ \int e^{ax} \cos x \, dx &= \frac{e^{ax}}{1 + a^2}(\sin x + a \cos x) + C \end{aligned}$$