Gaussian Integral

To compute the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

let's multiply

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy$$

Gaussian Integral

In polar coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \text{jacobian} \quad J &= r \\ \text{new volume element} \quad J\,drd\theta &= r\,drd\theta \end{aligned}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,drd\theta = \pi$$

Gaussian Integral

and the value of Gaussian integral is

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$