Solving the Integral $\int_{-\infty}^{\infty} e^{-ax^2}\cos(\mu x)\,dx$

Let

$$I(\mu) = \int_0^\infty e^{-x^2}\cos(\mu x)\,dx \tag{1}$$

Differentiation with respect to μ behind the integral sign gives:

$$I'(\mu) = -\int_0^\infty xe^{-x^2}\sin(\mu x)\,dx \tag{2}$$

The differentiation is legitimate because the resulting integral is uniformly convergent in μ.

Now we integrate by parts applying the formula:

$$\int u'v dx = uv - \int uv' dx \tag{3}$$

$$I'(\mu) = -\int_0^\infty xe^{-x^2}\sin(\mu x)\,dx = \tag{4}$$

$$\left| \begin{array}{rcl} -xe^{-x^2} = (\frac{1}{2}e^{-x^2})' & = & u' \\ \sin(\mu x) & = & v \end{array} \right| \left| \begin{array}{rcl} u & = & \frac{1}{2}e^{-x^2} \\ v' & = & \mu\cos(\mu x) \end{array} \right| =$$

$$\frac{1}{2}[e^{-x^2}\sin(\mu x)]_{x=0}^{x=\infty} - \frac{\mu}{2}\int_0^\infty e^{-x^2}\cos(\mu x)\,dx$$

and we get

$$I'(\mu) = -\frac{\mu}{2}\,I(\mu) \tag{5}$$

$$\frac{dI}{I} = -\frac{\mu}{2}\,d\mu \tag{6}$$

After integration we obtain

$$I(\mu) = C\,e^{-\mu^2/4} \tag{7}$$

To find a constant C we set $\mu = 0$. This gives

$$C = I(0) = \int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi} \tag{8}$$

and

$$I(\mu) = \int_0^\infty e^{-x^2}\cos(\mu x)\,dx = \frac{1}{2}\sqrt{\pi}\,e^{-\mu^2/4} \tag{9}$$

$$\int_0^\infty e^{-ax^2}\cos(\mu x)\,dx = \int_0^\infty e^{-(\sqrt{a}x)^2}\cos(\mu x)\,dx = \left| \begin{array}{rcl} \sqrt{a}\,x & = & w \\ \sqrt{a}\,dx & = & dw \end{array} \right| = \tag{10}$$

$$\frac{1}{\sqrt{a}}\int_0^\infty e^{-w^2}\cos(\frac{\mu}{\sqrt{a}}w)\,dw = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-\mu^2/4a}$$

and we have the integral

$$\int_{-\infty}^\infty e^{-ax^2}\cos(\mu x)\,dx = \sqrt{\frac{\pi}{a}}\,e^{-\mu^2/4a} \tag{11}$$