How to find the residue at a pole of order N

Suppose $f(z)$ has a simple pole at $z = a$. Then

$$f(z) = \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots \tag{1}$$

If we multiply both sides by $z - a$ and then let $z \to a$, we have

$$b_1 = \lim_{z \to a} (z-a) f(z) \quad \text{(simple pole)} \tag{2}$$

Suppose now that $f(z)$ has a pole of order N, as in equation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{N} \frac{b_n}{(z-a)^n} \tag{3}$$

Let us multiply the above equation by $(z-a)^N$

$$(z-a)^N f(z) = b_N + b_{N-1}(z-a) + \cdots + b_1 (z-a)^{N-1} + a_0 (z-a)^N + \cdots \tag{4}$$

The equation (4) represents the Taylor series expansion of $(z-a)^N f(z)$. If we let $G(z) = (z-a)^N f(z)$, then equation (4) can be written as

$$G(z) = G(a) + G'(a)(z-a) + \cdots + \frac{1}{(N-1)!} \left(\frac{d^{N-1} G}{dz^{N-1}} \right)_{z=a} (z-a)^{N-1} + \cdots \tag{5}$$

By comparing equation (5) with equation (4) we see that

$$b_1 = \frac{1}{(N-1)!} \lim_{z \to a} \left[\frac{d^{N-1} (z-a)^N f(z)}{dz^{N-1}} \right] \tag{6}$$

References

[1] McQuarrie, Donald A. (2015) *Mathematical Methods for Scientists and Engineers* Viva Books, page 915

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