Limits of Fresnel integrals calculated in an easy way

Limits of cosine and sine Fresnel integrals are

$$\int_0^\infty \cos(t^2)\,dt \tag{1}$$

and

$$\int_0^\infty \sin(t^2)\,dt \tag{2}$$

We notice that we may use Euler's formula

$$e^{-it^2} = \cos(t^2) - i\sin(t^2) \tag{3}$$

where $i = \sqrt{-1}$. We have

$$\int_{-\infty}^\infty e^{-it^2}\,dt = \int_{-\infty}^\infty \cos(t^2)\,dt - i\int_{-\infty}^\infty \sin(t^2)\,dt = \sqrt{\frac{\pi}{i}} \tag{4}$$

what is known from solving the Gaussian integral like $\int_{-\infty}^\infty e^{-ax^2} = \sqrt{\pi/a}$. We see that

$$\sqrt{\frac{\pi}{i}} = \sqrt{\frac{i}{i}\frac{\pi}{i}} = \sqrt{-i\pi} = \sqrt{-i}\sqrt{\pi} \tag{5}$$

We compute the $\sqrt{-i}$ as follows

$$\sqrt{-i} = (-i)^{1/2} \tag{6}$$

$$-i = \cos\left(\frac{3}{2}\pi + 2\pi m\right) + i\sin\left(\frac{3}{2}\pi + 2\pi m\right) = e^{i(\frac{3}{2}\pi+2\pi m)} \tag{7}$$

$$\sqrt{-i} = \left(e^{i(\frac{3}{2}\pi+2\pi m)}\right)^{1/2} = e^{i(\frac{3}{4}\pi+\pi m)} = \cos\left(\frac{3}{4}\pi+\pi m\right)+i\sin\left(\frac{3}{4}\pi+\pi m\right) \tag{8}$$

where $m = 0, 1$. Therefrom we receive two values r_1 and r_2 for the square root of the imaginary unit $-i$

$$r_1 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \tag{9}$$

$$r_2 = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \tag{10}$$

$$\sqrt{\frac{\pi}{i}} = \begin{cases} \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\sqrt{\pi} \\ \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\sqrt{\pi} \end{cases} \tag{11}$$

By comparing the coefficients in equation (4) and in equation (11) we find out that

$$\int_{-\infty}^{\infty} \cos(t^2)\, dt = \sqrt{\frac{\pi}{2}} \tag{12}$$

and

$$\int_{-\infty}^{\infty} \sin(t^2)\, dt = \sqrt{\frac{\pi}{2}} \tag{13}$$

Because the functions $\cos(t^2)$ and $\sin(t^2)$ are even, we receive for the limits of the Fresnel integrals

$$\int_{0}^{\infty} \cos(t^2)\, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{14}$$

and

$$\int_{0}^{\infty} \sin(t^2)\, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{15}$$

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