Fourier Transform of the Dirac Delta Function

The pair of Fourier transforms is

$$\hat{C}(k) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} C(x) e^{-ikx}\, dx \tag{1}$$

$$C(x) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \hat{C}(k) e^{ikx}\, dk \tag{2}$$

From the property of the Dirac delta function

$$f(x_0) = \int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx \tag{3}$$

setting $x_0 = 0$ we arrive at the Fourier transform of the Dirac delta function

$$\hat{\delta}(k) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx = \frac{1}{\sqrt{2\,\pi}} e^{-ik0} = \frac{1}{\sqrt{2\,\pi}} \tag{4}$$

Now if we invert the transform, we obtain the expression for the Dirac delta function

$$\delta(x) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \hat{\delta}(k) e^{ikx}\, dk = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk \tag{5}$$

and we also have that

$$\delta(x - x_0) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)}\, dk \tag{6}$$

From the equations (3) and (6) we derive the property

$$f(x_0) = \int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = \frac{1}{2\,\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) e^{ik(x-x_0)}\, dk\, dx \tag{7}$$