Extrema with constraints exercises

Exercise 1. Find the shape and the maximum volume of the paralellepiped, whose total surface area is A, using the method of Lagrange multipliers.

Solution: We have

$$V = abc \quad \text{with} \quad g = ab + bc + ac = A/2 \tag{1}$$

Equations of the Lagrange multipliers method are

$$\frac{\partial V}{\partial a} - \lambda \frac{\partial g}{\partial a} = bc - \lambda(b + c) = 0 \tag{2}$$
$$\frac{\partial V}{\partial b} - \lambda \frac{\partial g}{\partial b} = ac - \lambda(a + c) = 0$$
$$\frac{\partial V}{\partial c} - \lambda \frac{\partial g}{\partial b} = ab - \lambda(a + b) = 0$$

We solve the equations and find that $a = b = c = \sqrt{\frac{1}{3}(A/2)} = (A/6)^{\frac{1}{2}}$ and the volume of the cube is $(A/6)^{\frac{3}{2}}$.

Exercise 2. The equation

$$x + y + z = 1 \tag{3}$$

represents a plane that cuts the axes at the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Find the shortest distance from the origin to this plane.

Solution: We shall minimize $d^2 = x^2 + y^2 + z^2$ with constraint $g = x + y + z = 1$. Equations of the Lagrange multipliers method are

$$\frac{\partial d^2}{\partial x} - \lambda \frac{\partial g}{\partial x} = 2x - \lambda = 0 \tag{4}$$
$$\frac{\partial d^2}{\partial y} - \lambda \frac{\partial g}{\partial y} = 2y - \lambda = 0 \tag{5}$$
$$\frac{\partial d^2}{\partial z} - \lambda \frac{\partial g}{\partial z} = 2z - \lambda = 0 \tag{6}$$

This gives $x = y = z$. Substituting this result into $x + y + z = 1$ gives $x = y = z = 1/3$, and $d = 1/\sqrt{3}$.

References

[1] Donald A. McQuarrie (2015) *Mathematical Methods for Scientists and Engineers* University Science Books

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