Note on extrema of $f(x, y)$

The short form of the Taylor formula for the function f of two variables x and y is:

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_0 h + \left(\frac{\partial f}{\partial y}\right)_0 k + \tag{1}$$
$$+\frac{1}{1 \cdot 2}\left[\left(\frac{\partial^2 f}{\partial x^2}\right)_0 h^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 hk + \left(\frac{\partial^2 f}{\partial y^2}\right)_0 k^2\right] + R_3$$

where

$$R_3 = \frac{1}{1 \cdot 2 \cdot 3}\left[\frac{\partial^3 f}{\partial x^3}h^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}h^2 k + 3\frac{\partial^3 f}{\partial x \partial y^2}hk^2 + \frac{\partial^3 f}{\partial y^3}k^3\right] \tag{2}$$

In R_3 all derivatives are being computed at $x = x_0 + \theta h$ and $y = y_0 + \theta k$ where $0 < \theta < 1$. Introducing the short notation

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_0 = A, \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 = B, \quad \left(\frac{\partial^2 f}{\partial y^2}\right)_0 = C \tag{3}$$

The equation (1) can be rewritten as

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_0 h + \left(\frac{\partial f}{\partial y}\right)_0 k + \tag{4}$$
$$+\frac{1}{1 \cdot 2}\left[Ah^2 + 2Bhk + Ck^2\right] + R_3$$

We say that the function $f(x, y)$ of two variables has a maximum at point (x_0, y_0) if in a certain neighbourhood of the radius $\rho > 0$ the value of the function $f(x_0, y_0)$ is greater than the value of the function at any other point (x, y) of this neighbourhood. The definition of a minimum of the function $f(x, y)$ of two variables is similar.

Theorem. If the function $z = f(x, y)$ has derivatives

$$\left(\frac{\partial f}{\partial x}\right)_0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_0 \tag{5}$$

at point $M(x_0, y_0)$, and if it has at this point a maximum or a minimum then

$$\left(\frac{\partial f}{\partial x}\right)_0 = \left(\frac{\partial f}{\partial y}\right)_0 = 0 \tag{6}$$

Indeed, if the function $f(x, y)$ has maximum at point $M(x_0, y_0)$, then the function of one variable $f(x, y_0)$ has maximum at the point x_0 and then it should be $f'_x(x_0, y_0) = 0$. In a similar way we can prove that $f'_y(x_0, y_0) = 0$.

Let us assume that at the point $M(x_0, y_0)$ both partial first order derivatives of the function $z = f(x, y)$ are equal zero. Then the Taylor formula (4) takes the form

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = \frac{1}{1 \cdot 2}\left[Ah^2 + Bhk + Ck^2\right] + R_3 \tag{7}$$

Let us write

$$T(h, k) = k^2\left[A\left(\frac{h}{k}\right)^2 + 2B\left(\frac{h}{k}\right) + C\right] \tag{8}$$

If we denote $h/k = z$

$$T(h, k) = k^2(Az^2 + 2Bz + C) \tag{9}$$

The discriminant of the equation (9) is $\Delta = 4B^2 - 4AC$. If $B^2 - AC < 0$ then z does not change its sign. Then the trinomial $T(h, k)$ and with it the difference $f(x_0 + h, y_0 + k) - f(x_0, y_0)$ do not change their sign, and it is positive if $A > 0$ and negative if $A < 0$. This means that if

$$\delta = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_0^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)_0 \left(\frac{\partial^2 f}{\partial y^2}\right)_0 < 0 \text{ and } \left(\frac{\partial^2 f}{\partial x^2}\right)_0 < 0 \tag{10}$$

then at point $M(x_0, y_0)$ there is a maximum, and if

$$\delta = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_0^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)_0 \left(\frac{\partial^2 f}{\partial y^2}\right)_0 < 0 \text{ and } \left(\frac{\partial^2 f}{\partial x^2}\right)_0 > 0 \tag{11}$$

then at point $M(x_0, y_0)$ there is a minimum.

Example. Let us have given the function of two variables

$$z = x^2 + y^2 - 2x + 6y \tag{12}$$

The derivatives

$$\frac{\partial z}{\partial x} = 2x - 2 \text{ and } \frac{\partial z}{\partial y} = 2y + 6 \tag{13}$$

are equal zero at the point $M(1, -3)$.

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)_0 = 0 \qquad \left(\frac{\partial^2 z}{\partial x^2}\right)_0 = 2 \qquad \left(\frac{\partial^2 z}{\partial y^2}\right)_0 = 2 \tag{14}$$

Because

$$\delta = -4 < 0 \qquad \text{and} \qquad \left(\frac{\partial^2 z}{\partial x^2}\right)_0 = 2 > 0 \tag{15}$$

then at the point $M(1, -3)$ there is a minimum.

Exercises

1. Find the extrema of the function $z = 2xy - 3x^2 - 2y^2 + 10$.
2. Find the extrema of the function $z = x^2 + xy + y^2 + x - y + 1$.

Summary Let $f(x, y)$ be continuous and let all the second-order partial derivatives be continuous throughout some neighborhood of (a, b). Furthermore, let $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f^2_{xy}(a, b)$. Some authors call D the Hessian determinant of $f(x, y)$. If

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0 \tag{16}$$

then $f(a, b)$ will be

1. a local maximum if $f_{xx}(a, b) < 0$ and $D(a, b) > 0$
2. a local minimum if $f_{xx}(a, b) > 0$ and $D(a, b) > 0$
3. a saddle point if $D(a, b) < 0$
4. if $D(a, b) = 0$, higher partial derivatives must be considered

References

[1] Romanowski, Ś., Wrona W. (1968) *Matematyka wyższa dla studiów technicznych, Part I* PWN Warszawa

[2] Donald A. McQuarrie (2015) *Mathematical Methods for Scientists and Engineers* University Science Books

Pawel Jan Piskorz (paweljs@gmail.com)