Note on Euler formula

We have the Maclaurin series of the function e^x

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{1}$$

The imaginary unit i has the property that $i^2 = -1$. Let us expand the function e^{ix} in the Maclaurin series. We may write then

$$\begin{aligned} e^{ix} = \sum_{n=0}^{\infty} \frac{1}{n!}(ix)^n = \frac{1}{0!}(ix)^0 + \frac{1}{1!}(ix)^1 + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 \\ + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \frac{1}{6!}(ix)^6 + \frac{1}{7!}(ix)^7 + \dots \\ = \frac{1}{0!} + i\frac{1}{1!}x - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + i\frac{1}{5!}x^5 - \frac{1}{6!}x^6 - i\frac{1}{7!}x^7 + \dots \end{aligned} \tag{2}$$

Now in the equation (2) we have terms without i and with i and we can rewrite the Maclaurin series of e^{ix} as

$$\begin{aligned} e^{ix} = \frac{1}{0!} - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \\ + i(\frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots) \end{aligned} \tag{3}$$

We recognize in equation (3) the patterns for expansion of functions $\cos(x)$ and $i\sin(x)$ in Maclaurin series. Therefore we have the relation

$$e^{ix} = \cos(x) + i\sin(x) \tag{4}$$

which is known as the Euler formula.

Based on equation (4) we have the formulas

$$e^{i\pi} = -1 \tag{5}$$

or

$$e^{i\pi} + 1 = 0 \tag{6}$$

The above equation (6) joins the numbers e, i, π, 1 and 0 in an interesting way.

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