Euler gamma function product formula

The Euler gamma function is defined as

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt \tag{1}$$

Let us substitute $t = u^2$. Then $t^{p-1} = (u^2)^{p-1} = u^{2p-2}$

$$\begin{aligned}\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt = \left| \begin{array}{ccc} t & = & u^2 \\ dt & = & 2udu \end{array} \right| &= 2\int_0^\infty u^{2p-2} e^{-u^2} u\, du \\ &= 2\int_0^\infty u^{2p-1} e^{-u^2}\, du\end{aligned} \tag{2}$$

$$\begin{aligned}\Gamma(1-p) = 2\int_0^\infty v^{2(1-p)-1} e^{-v^2}\, dv &= 2\int_0^\infty v^{1-2p} e^{-v^2}\, dv \\ &= 2\int_0^\infty v^{-(2p-1)} e^{-v^2}\, dv\end{aligned} \tag{3}$$

Now

$$\Gamma(p)\Gamma(1-p) = 4\int_0^\infty \int_0^\infty \left(\frac{u}{v}\right)^{2p-1} e^{-(u^2+v^2)}\, dudv \tag{4}$$

We change the u and v variables into polar coordinates ρ and Θ as follows

$$\begin{aligned} u &= \rho\cos\Theta \\ u &= \rho\sin\Theta \end{aligned}$$

what gives

$$\begin{aligned} u^2 + v^2 &= \rho^2 \\ \frac{u}{v} &= \operatorname{ctg}\Theta \end{aligned}$$

with Jacobian ρ

$$\begin{aligned}\Gamma(p)\Gamma(1-p) &= 4\int_0^\infty \int_0^{\pi/2} \operatorname{ctg}^{2p-1}\Theta e^{-\rho^2} \rho\, d\rho d\Theta \\ &= 4\int_0^{\pi/2} \operatorname{ctg}^{2p-1}\Theta\, d\Theta \cdot \int_0^\infty e^{-\rho^2}\rho\, d\rho \\ &= 2\int_0^{\pi/2} \operatorname{ctg}^{2p-1}\Theta\, d\Theta\end{aligned} \tag{5}$$

We substitute

$$\begin{aligned} \operatorname{ctg}\Theta &= \sqrt{x} \\ \Theta &= \operatorname{arcctg}\sqrt{x} \end{aligned} \tag{6}$$

and we have

$$\frac{d\Theta}{dx} = -\frac{1}{(\sqrt{x})^2+1}\cdot\frac{1}{2\sqrt{x}} \tag{7}$$

$$\operatorname{ctg}^{2p-1}\Theta = (\sqrt{x})^{2p-1} = \frac{x^p}{\sqrt{x}} \tag{8}$$

We have the following limits of integration

$$\text{for} \quad \Theta = 0 \ \ \sqrt{x} = \operatorname{ctg} 0 = +\infty \quad \text{and} \quad x = \infty \tag{9}$$

$$\text{for} \quad \Theta = \frac{\pi}{2} \ \ \sqrt{x} = \operatorname{ctg}\frac{\pi}{2} = 0 \quad \text{and} \quad x = 0 \tag{10}$$

$$\Gamma(p)\Gamma(1-p) = 2\int_{\infty}^{0} \frac{x^p}{\sqrt{x}}\cdot\frac{-1}{2\sqrt{x}}\cdot\frac{1}{x+1}\,dx \tag{11}$$

$$= \int_0^{\infty} \frac{x^p}{x(x+1)}\,dx = \int_0^{\infty}\frac{x^{p-1}}{x+1}\,dx = \frac{\pi}{\sin p\pi}$$

References

[1] Smirnov, W.I., (1965) *Matematyka wyższa, Tom III, Część druga* page 249, PWN Warszawa

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