On the concept of divergence

Vector $\vec{F}(x,y,z)$ is a function of coordinates x, y, z and it describes a vector field of flow in three dimensions

$$\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k} \tag{1}$$

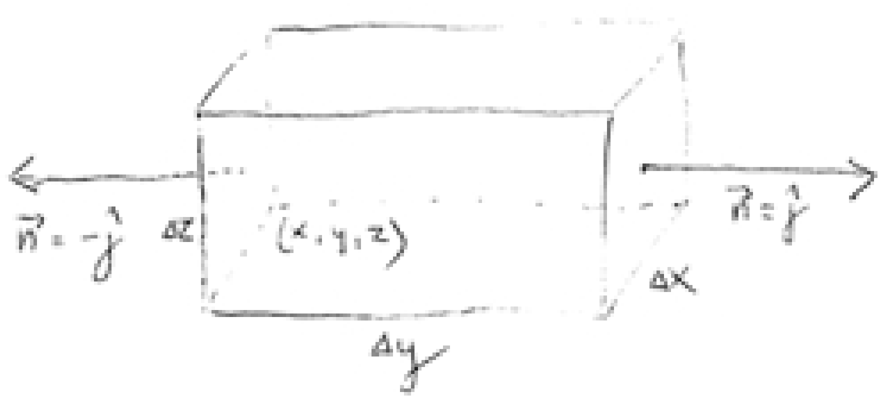

Figure 1: Flow out of the box placed at the point (x, y, z) through the faces perpendicular to the Oy axis

Let $\vec{F}$ be the amount of flow per unit time per unit area normal to the flow. Let us consider a rectangular box defined by increments $\Delta x, \Delta y, \Delta z$ from point (x, y, z). Vectors $\vec{n}$ are normals to the faces of the box pointing outward. The flow out from the box through the left hand face is

$$\vec{F} \cdot (-\hat{j})\Delta x \Delta z = -Q(x,y,z)\Delta x\,\Delta z \tag{2}$$

and through the right hand face is

$$\vec{F} \cdot \hat{j}\Delta x\,\Delta z = Q(x, y+\Delta y, z)\Delta x\,\Delta z \tag{3}$$

The total flow out of the box through these two faces is then

$$\begin{aligned}[Q(x,y+\Delta y,z) - Q(x,y,z)]\Delta x\,\Delta z &= \frac{\partial Q(x,y,z)}{\partial y}\Delta x\,\Delta y\,\Delta z \\ &= Q_y(x,y,z)\Delta x\,\Delta y\,\Delta z\end{aligned} \tag{4}$$

If similarly we consider the flow out of the box through the other faces we receive

$$[P_x(x,y,z) + Q_y(x,y,z) + R_z(x,y,z)]\Delta x\,\Delta y\,\Delta z \tag{5}$$

Dividing by the volume $\Delta x \Delta y \Delta z$ and passing to the limit as dimensions of the box tend to zero yields

$$P_x(x,y,z) + Q_y(x,y,z) + R_z(x,y,z) \tag{6}$$

as the flow out at the point (x,y,z) from the volume element $dx\,dy\,dz$ per volume element $dx\,dy\,dz$ per unit time. The flow out from the volume element $dx\,dy\,dz$ per volume element $dx\,dy\,dz$ per unit time is called divergence

$$\begin{aligned} div\vec{F} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(P\hat{i} + Q\hat{j} + R\hat{k}\right) \\ &= \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \end{aligned} \tag{7}$$

Let us change the Cartesian coordinates from x,y,z to u,v,w and have

$$\vec{F}(u,v,w) = L(u,v,w)\hat{i}' + M(u,v,w)\hat{j}' + N(u,v,w)\hat{k}' \tag{8}$$

Let us assume that $F(x,y,z) = F(u,v,w)$. Then

$$L(u,v,w) = \vec{F} \cdot \hat{i}' = P(\hat{i} \cdot \hat{i}') + Q(\hat{j} \cdot \hat{i}') + R(\hat{k} \cdot \hat{i}') \tag{9}$$

$$M(u,v,w) = \vec{F} \cdot \hat{j}' = P(\hat{i} \cdot \hat{j}') + Q(\hat{j} \cdot \hat{j}') + R(\hat{k} \cdot \hat{j}') \tag{10}$$

$$N(u,v,w) = \vec{F} \cdot \hat{k}' = P(\hat{i} \cdot \hat{k}') + Q(\hat{j} \cdot \hat{k}') + R(\hat{k} \cdot \hat{k}') \tag{11}$$

We can express the divergence of $\vec{F}$ as

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{12}$$

or as

$$\frac{\partial L}{\partial u} + \frac{\partial M}{\partial v} + \frac{\partial N}{\partial w} \tag{13}$$

We will show that the two expressions in equations (12) and (13) for divergence $\nabla \cdot \vec{F}$ in Cartesian coordinates x,y,z and u,v,w are equal. We have the gradients

$$\nabla P = \frac{\partial P}{\partial x}\hat{i} + \frac{\partial P}{\partial y}\hat{j} + \frac{\partial P}{\partial z}\hat{k} \tag{14}$$

$$\nabla P = \frac{\partial P}{\partial u}\hat{i}' + \frac{\partial P}{\partial v}\hat{j}' + \frac{\partial P}{\partial w}\hat{k}' \tag{15}$$

$$\nabla Q = \frac{\partial Q}{\partial x}\hat{i} + \frac{\partial Q}{\partial y}\hat{j} + \frac{\partial Q}{\partial z}\hat{k} \tag{16}$$

$$\nabla Q = \frac{\partial Q}{\partial u}\hat{i}' + \frac{\partial Q}{\partial v}\hat{j}' + \frac{\partial Q}{\partial w}\hat{k}' \tag{17}$$

$$\nabla R = \frac{\partial R}{\partial x}\hat{i} + \frac{\partial R}{\partial y}\hat{j} + \frac{\partial R}{\partial z}\hat{k} \tag{18}$$

$$\nabla R = \frac{\partial R}{\partial u}\hat{i}' + \frac{\partial R}{\partial v}\hat{j}' + \frac{\partial R}{\partial w}\hat{k}' \tag{19}$$

The following relations are valid based on equation (15)

$$\nabla P \cdot \hat{i}' = \frac{\partial P}{\partial u} \tag{20}$$

$$\nabla P \cdot \hat{j}' = \frac{\partial P}{\partial v} \tag{21}$$

$$\nabla P \cdot \hat{k}' = \frac{\partial P}{\partial w} \tag{22}$$

on equation (17)

$$\nabla Q \cdot \hat{i}' = \frac{\partial Q}{\partial u} \tag{23}$$

$$\nabla Q \cdot \hat{j}' = \frac{\partial Q}{\partial v} \tag{24}$$

$$\nabla Q \cdot \hat{k}' = \frac{\partial Q}{\partial w} \tag{25}$$

and on equation (19)

$$\nabla R \cdot \hat{i}' = \frac{\partial R}{\partial u} \tag{26}$$

$$\nabla R \cdot \hat{j}' = \frac{\partial R}{\partial v} \tag{27}$$

$$\nabla R \cdot \hat{k}' = \frac{\partial R}{\partial w} \tag{28}$$

Taking into account the equation (9) we can write

$$\frac{\partial L}{\partial u} = \frac{\partial P}{\partial u}(\hat{i} \cdot \hat{i}') + \frac{\partial Q}{\partial u}(\hat{j} \cdot \hat{i}') + \frac{\partial R}{\partial u}(\hat{k} \cdot \hat{i}') \tag{29}$$

Similarly based on equation (10) we have

$$\frac{\partial M}{\partial v} = \frac{\partial P}{\partial v}(\hat{i} \cdot \hat{j}') + \frac{\partial Q}{\partial v}(\hat{j} \cdot \hat{j}') + \frac{\partial R}{\partial v}(\hat{k} \cdot \hat{j}') \tag{30}$$

and based on equation (11) we receive

$$\frac{\partial N}{\partial w} = \frac{\partial P}{\partial w}(\hat{i} \cdot \hat{k}') + \frac{\partial Q}{\partial w}(\hat{j} \cdot \hat{k}') + \frac{\partial R}{\partial w}(\hat{k} \cdot \hat{k}') \tag{31}$$

Using relations (20, 23, 26) we can rewrite equation (29) as

$$\frac{\partial L}{\partial u} = [(\nabla P \cdot \hat{i}')\hat{i}'] \cdot \hat{i} + [(\nabla Q \cdot \hat{i}')\hat{i}'] \cdot \hat{j} + [(\nabla R \cdot \hat{i}')\hat{i}'] \cdot \hat{k} \tag{32}$$

By analogy using relations (21, 24, 27) the equation (30) can be rewritten as

$$\frac{\partial M}{\partial v} = [(\nabla P \cdot \hat{j}')\hat{j}'] \cdot \hat{i} + [(\nabla Q \cdot \hat{j}')\hat{j}'] \cdot \hat{j} + [(\nabla R \cdot \hat{j}')\hat{j}'] \cdot \hat{k} \tag{33}$$

and using relations (22, 25, 28) the equation (31) can be rewritten as

$$\frac{\partial N}{\partial w} = [(\nabla P \cdot \hat{k}')\hat{k}'] \cdot \hat{i} + [(\nabla Q \cdot \hat{k}')\hat{k}'] \cdot \hat{j} + [(\nabla R \cdot \hat{k}')\hat{k}'] \cdot \hat{k} \tag{34}$$

The equations (32, 33, 34) can be rewritten as

$$\frac{\partial L}{\partial u} = \left[\frac{\partial P}{\partial u}\hat{i}'\right] \cdot \hat{i} + \left[\frac{\partial Q}{\partial u}\hat{i}'\right] \cdot \hat{j} + \left[\frac{\partial R}{\partial u}\hat{i}'\right] \cdot \hat{k} \tag{35}$$

$$\frac{\partial M}{\partial v} = \left[\frac{\partial P}{\partial v}\hat{j}'\right] \cdot \hat{i} + \left[\frac{\partial Q}{\partial v}\hat{j}'\right] \cdot \hat{j} + \left[\frac{\partial R}{\partial v}\hat{j}'\right] \cdot \hat{k} \tag{36}$$

$$\frac{\partial N}{\partial w} = \left[\frac{\partial P}{\partial w}\hat{k}'\right] \cdot \hat{i} + \left[\frac{\partial Q}{\partial w}\hat{k}'\right] \cdot \hat{j} + \left[\frac{\partial R}{\partial w}\hat{k}'\right] \cdot \hat{k} \tag{37}$$

Now we add expressions on the left hand side and right hand side of the equations (35, 36, 37) obtaining

$$\frac{\partial L}{\partial u} + \frac{\partial M}{\partial v} + \frac{\partial N}{\partial w} = (\nabla P) \cdot \hat{i} + (\nabla Q) \cdot \hat{j} + (\nabla R) \cdot \hat{k} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{38}$$

The above equation states that the two expressions (12) and (13) for divergence $\nabla \cdot \vec{F}$ in Cartesian coordinates x, y, z and u, v, w are indeed equal.

References

[1] Lindgren, B.W. (1963) *Vector Calculus* The Macmillan Company, New York, Collier-Macmillan Limited, London

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