On the origin of curl or rotation

Let us calculate the line integral of a vector field $\vec{V}(x, y, z)$ around the closed path C_1, C_2, C_3, and C_4 shown in Figure 1.

$$\oint_C d\vec{r}\vec{V} \tag{1}$$

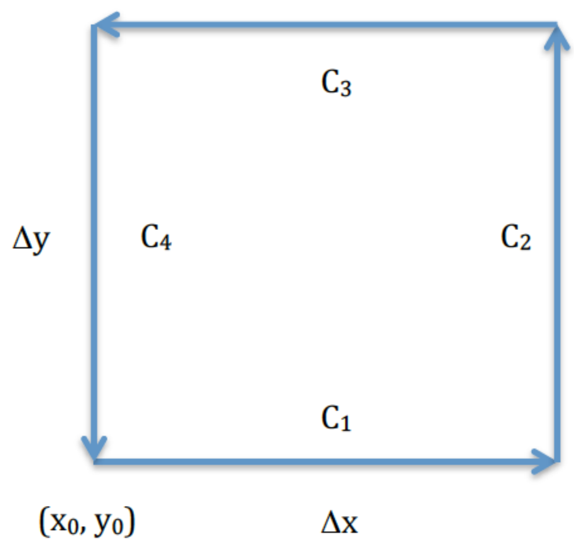

Figure 1: Closed path of integration for curl definition

Let us imagine a vector field $\vec{V}$ on a plane. Let the vector $\vec{V}$ has the component V_x along the Ox axis and the component V_y along the Oy axis in the rectangular Cartesian system of coordinates.

The starting point of the integration is the point (x_0, y_0).

The integral along the closed path C_1, C_2, C_3, and C_4 can be broken up into four parts. First consider the integration along C_1, where $y = y_0$ and x varies from x_0 to $x_0 + \Delta x$:

$$\int_{C_1} d\vec{r}\vec{V} = \int_{x_0}^{x_0+\Delta x} dx\, V_x \tag{2}$$

Along this segment we can expand $V_x(x, y_0)$ with a Taylor series, keeping up to the linear term in x:

$$V_x(x, y_0) \approx V_x(x_0, y_0) + \left.\frac{\partial V_x}{\partial x}\right|_{(x_0,y_0)} (x - x_0) \tag{3}$$

We have then

$$\int_{C_1} d\vec{r}\vec{V} = \int_{x_0}^{x_0+\Delta x} dx\, V_x = \tag{4}$$

$$= \int_{x_0}^{x_0+\Delta x} dx\, V_x(x_0, y_0) + \int_{x_0}^{x_0+\Delta x} dx \left(\left.\frac{\partial V_x}{\partial x}\right|_{(x_0,y_0)} (x - x_0) \right)$$

We compute

$$\int_{x_0}^{x_0+\Delta x} dx\, V_x(x_0, y_0) = V_x(x_0, y_0) \int_{x_0}^{x_0+\Delta x} dx = V_x(x_0, y_0)\Delta x \tag{5}$$

and

$$\int_{x_0}^{x_0+\Delta x} dx \left(\left.\frac{\partial V_x}{\partial x}\right|_{(x_0,y_0)} (x - x_0) \right) = \left.\frac{\partial V_x}{\partial x}\right|_{(x_0,y_0)} \int_{x_0}^{x_0+\Delta x} dx\, (x - x_0) \tag{6}$$

We can substitute $x - x_0 = u$ and we have $dx = du$. The integration lower limit for $x = x_0$ is $u = x_0 - x_0 = 0$ and for $x = x_0 + \Delta x$ we have the integration upper limit $u = x - x_0 = x_0 + \Delta x - x_0 = \Delta x$

$$\int_{x_0}^{x_0+\Delta x} dx\, (x - x_0) = \int_0^{\Delta x} du\, u = \left[\frac{u^2}{2} \right]_0^{\Delta x} = \frac{(\Delta x)^2}{2} \tag{7}$$

After substituting we have for the path C_1

$$\int_{C_1} d\vec{r}\vec{V} = \int_{x_0}^{x_0+\Delta x} dx\, V_x \approx V_x(x_0, y_0)\Delta x + \frac{1}{2} \left.\frac{\partial V_x}{\partial x}\right|_{(x_0,y_0)} (\Delta x)^2 \tag{8}$$

Let us now integrate along the path C_3.

$$\int_{C_3} d\vec{r}\,\vec{V} = \int_{x_0+\Delta x}^{x_0} dx\, V_x \tag{9}$$

We expand

$$V_x(x, y_0+\Delta y) = V_x(x_0, y_0) + \left.\frac{\partial V_x}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \tag{10}$$

$$+\left.\frac{\partial V_x}{\partial y}\right|_{(x_0,y_0)} (y_0 + \Delta y - y_0)$$

and we receive

$$V_x(x, y_0+\Delta y) = \tag{11}$$

$$= V_x(x_0, y_0) + \left.\frac{\partial V_x}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial V_x}{\partial y}\right|_{(x_0,y_0)} \Delta y$$

We see that

$$\int_{C_3} d\vec{r}\,\vec{V} = \int_{x_0+\Delta x}^{x_0} dx\, V_x = -\int_{x_0}^{x_0+\Delta x} dx\, V_x = \tag{12}$$

$$= -V_x(x_0, y_0)\Delta x - \frac{1}{2}\left.\frac{\partial V_x}{\partial x}\right|_{(x_0,y_0)} (\Delta x)^2 - \left.\frac{\partial V_x}{\partial y}\right|_{(x_0,y_0)} \Delta x \Delta y$$

We have for the sum

$$\int_{C_1} d\vec{r}\,\vec{V} + \int_{C_3} d\vec{r}\,\vec{V} = \tag{13}$$

$$= \int_{x_0}^{x_0+\Delta x} dx\, V_x(x, y_0) + \int_{x_0+\Delta x}^{x_0} dx\, V_x(x, y_0+\Delta y) = -\left.\frac{\partial V_x}{\partial y}\right|_{(x_0,y_0)} \Delta x \Delta y$$

We notice that we can exchange symbol x with symbol y and symbol x_0 with symbol y_0 in equation (13) and obtain

$$-\int_{C_4} d\vec{r}\,\vec{V} - \int_{C_2} d\vec{r}\,\vec{V} = \tag{14}$$

$$= \int_{y_0}^{y_0+\Delta y} dy\, V_y(x_0, y) + \int_{y_0+\Delta y}^{y_0} dy\, V_y(x_0+\Delta x, y) = -\left.\frac{\partial V_y}{\partial x}\right|_{(x_0,y_0)} \Delta x \Delta y$$

so we have

$$\int_{C_4} d\vec{r}\,\vec{V} + \int_{C_2} d\vec{r}\,\vec{V} = \left.\frac{\partial V_y}{\partial x}\right|_{(x_0,y_0)} \Delta x \Delta y \tag{15}$$

We can summarize that

$$\begin{aligned}\oint_C d\vec{r}\,\vec{V} &= \int_{C_1} d\vec{r}\,\vec{V} + \int_{C_2} d\vec{r}\,\vec{V} + \int_{C_3} d\vec{r}\,\vec{V} + \int_{C_4} d\vec{r}\,\vec{V} \approx \\ &\approx \left.\frac{\partial V_y}{\partial x}\right|_{(x_0,y_0)} \Delta x \Delta y - \left.\frac{\partial V_x}{\partial y}\right|_{(x_0,y_0)} \Delta x \Delta y = \\ &= \left(\left.\frac{\partial V_y}{\partial x}\right|_{(x_0,y_0)} - \left.\frac{\partial V_x}{\partial y}\right|_{(x_0,y_0)} \right) \Delta x \Delta y\end{aligned} \tag{16}$$

The component of the curl of the vector field $\vec{V}(x,y,z)$ in the direction of the Oz axis of the Cartesian system of coordinates $Oxyx$ is defined as

$$(\text{curl } \vec{V}(x,y,z))_z = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \tag{17}$$

References

[1] Kusse, Bruce R. and Westwig, Erik A. (2006) *Mathematical Physics Applied Mathematics for Scientists and Engineers* 2nd Edition, WILEY-VCH Verlag GmbH & Co. KGaA

Pawel Jan Piskorz (paweljs@gmail.com)