Chebyshev Inequality

Let $\bar{x}$ denotes the average value of the random variable $\boldsymbol{x}$. We would like to learn what we could say about the probability that

$$\boldsymbol{x} \leq \bar{x} - \epsilon$$

and

$$\boldsymbol{x} \geq \bar{x} + \epsilon$$

i.e. that random variable $\boldsymbol{x}$ assumes values outside the range $\bar{x} \pm \epsilon$ where $\epsilon > 0$. Then $(\boldsymbol{x} - \bar{x})^2 \geq \epsilon^2$.

We have that the variance $\operatorname{Var} \boldsymbol{x}$ is

$$\begin{aligned} \operatorname{Var} \boldsymbol{x} &= \int_{-\infty}^{\infty} (\boldsymbol{x} - \bar{x})^2 f_{\boldsymbol{x}}(x)\, dx \qquad (1) \\ &\geq \int_{-\infty}^{\bar{x}-\epsilon} (\boldsymbol{x} - \bar{x})^2 f_{\boldsymbol{x}}(x)\, dx + \int_{\bar{x}+\epsilon}^{\infty} (\boldsymbol{x} - \bar{x})^2 f_{\boldsymbol{x}}(x)\, dx \\ &\geq \int_{-\infty}^{\bar{x}-\epsilon} \epsilon^2 f_{\boldsymbol{x}}(x)\, dx + \int_{\bar{x}+\epsilon}^{\infty} \epsilon^2 f_{\boldsymbol{x}}(x)\, dx \\ &= \epsilon^2 \left(\int_{-\infty}^{\bar{x}-\epsilon} f_{\boldsymbol{x}}(x)\, dx + \int_{\bar{x}+\epsilon}^{\infty} f_{\boldsymbol{x}}(x)\, dx \right) = \epsilon^2 \Pr(|\boldsymbol{x} - \bar{x}| \geq \epsilon) \end{aligned}$$

We have above that the probability $\Pr(|\boldsymbol{x} - \bar{x}| \geq \epsilon)$ that the random variable $\boldsymbol{x}$ assumes values outside the range $\bar{x} \pm \epsilon$

$$\Pr(|\boldsymbol{x} - \bar{x}| \geq \epsilon) = \int_{-\infty}^{\bar{x}-\epsilon} f_{\boldsymbol{x}}(x)\, dx + \int_{\bar{x}+\epsilon}^{\infty} f_{\boldsymbol{x}}(x)\, dx \qquad (2)$$

is always less than $\operatorname{Var} \boldsymbol{x} / \epsilon^2$

$$\Pr(|\boldsymbol{x} - \bar{x}| \geq \epsilon) \leq \frac{\operatorname{Var} \boldsymbol{x}}{\epsilon^2} \qquad (3)$$

References

[1] Helstrom, Carl W. (1984) *Probability and stochastic processes for engineers* Macmillan Publishing Company New York, Collier Macmillan Publishers London

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