Note on random variable probability density distribution characteristic function

The characteristic function $\Phi_{\mathbf{x}}(\omega)$ of the random variable $\mathbf{x}$ is, except for the sign of imaginary unit j, the Fourier transform of its probability density function,

$$\begin{aligned}\Phi_{\mathbf{x}}(\omega) &= E(e^{j\omega\mathbf{x}}) = E(\cos\omega\mathbf{x}) + jE(\sin\omega\mathbf{x}) \\ &= \int_{-\infty}^{\infty} f_{\mathbf{x}}(x) e^{j\omega x}\, dx\end{aligned} \tag{1}$$

where E denotes the expected value.

The characteristic function $\Phi_{\mathbf{x}}(\omega)$ can be expanded in Taylor series

$$\Phi_{\mathbf{x}}(\omega) = \sum_{k=0}^{\infty} \frac{(j\omega)^k}{k!} E(\mathbf{x}^k) \tag{2}$$

Knowing the characteristic function, we can find the probability density function by the inverse Fourier transformation,

$$f_{\mathbf{x}}(x) = \int_{-\infty}^{\infty} \Phi_{\mathbf{x}}(\omega) e^{-j\omega x}\, \frac{d\omega}{2\pi} \tag{3}$$

References

[1] Carl W. Helstrom, *Probability and Stochastic Processes for Engineers* Macmillan Publishing Company, New York, Collier Macmillan Publishers, London, 1984.

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